EM Wave Propagation in a Medium with Anisotropic Dielectric and Magnetic Tensors

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Abstract: Dependence of phase velocity on propagation direction is shown to be decoupled from birefringence. The latter is propagation difference for two polarizations; it depends on anisotropy of impedance tensor: square root of mu / epsilon.

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Propagation of waves in anisotropic media is favorite topic of research in Optics since A. Fresnel. If the medium is spatially homogeneous, temporally stationary and dielectric response is generally anisotropic, but linear, then plane wave solutions are well-studied, see *e.g.* [1, 2]. Modern day research in metamaterials, [3-5], puts forward the problem of propagation of EM waves in a homogeneous, stationary, linear medium, where dielectric susceptibility tensor and magnetic permeability tensor are both different from their vacuum values and generally anisotropic. The talk is devoted to the solution of this problem.

Maxwell equations and material relationships are taken as

$$\partial\mathbf{B}/\partial t = -\text{curl}\,\mathbf{E}, \quad \partial\mathbf{D}/\partial t = \text{curl}\,\mathbf{H}\,, \quad B_j = \mu_{jk}H_k, \quad D_j = \epsilon_{jk}E_k. \tag{1}$$

Monochromatic plane wave solution is assumed to have the dependence $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$ for each Cartesian field component. Then complex amplitudes of **E** and **H** satisfy the equations

$$-\mathbf{k}\times\mathbf{E} + \omega\hat{\mu}\mathbf{H} = \mathbf{0}, \qquad \omega\hat{\varepsilon}\mathbf{E} + \mathbf{k}\times\mathbf{H} = 0. \tag{2}$$

Consider symmetric tensor $\hat{\varepsilon}$ which, after division by ε_{vac}, has real dimensionless components, $\epsilon_{xx} = A$, $\epsilon_{yy} = B, \epsilon_{zz} = C$, $\epsilon_{xy} = \epsilon_{yx} = P, \epsilon_{xz} = \epsilon_{zx} = S$, $\epsilon_{zy} = \epsilon_{yz} = R$, and similar notations for tensor μ_{ij} divided by μ_{vac}, but with small letters a, b, c, p, s, r instead of capital ones. The equations (2) may be considered as a system of 6 linear equations for 6 Cartesian components, E_x, E_y, E_z and H_x, H_y, H_z, with zero right-hand-side. Non-zero solution of that system exists if and only if the determinant of matrix of the coefficients for that system is zero. Formally this determinant yields polynomial of 6-th power in Cartesian components k_x, k_y, k_z. However, time-reversibility of the systems (1, 2) leads to this statement: if some vector **k** is a solution of that equation (root of the polynomial), then vector $(-\mathbf{k})$ is a solution as well. Hence this polynomial must contain only the terms with even number of k_x, k_y, k_z-components. Moreover, for the given direction $\mathbf{m} = \mathbf{k}/|\mathbf{k}|$ the length $|\mathbf{k}|$ turns out to be a root of bi-quadratic equation, in agreement with the existence of only two linearly-independent types of polarization. We managed to right down that bi-quadratic equation, and hence its solution for general case. The corresponding formulae are very heavy, and in the talk we will present particular cases.

Below we establish surprising properties of those solutions. The formula

$$\widetilde{\mathrm{T}}[(\mathrm{T}\mathbf{A})\times(\mathrm{T}\mathbf{B})] = [\mathbf{A}\times\mathbf{B}]\cdot\mathrm{Det}(\mathrm{T}) \tag{3}$$

is valid for an arbitrary pair of vectors **A** and **B** in 3D-space and arbitrary non-degenerate 3x3 matrix T, where $\widetilde{\mathrm{T}}$ denotes transpose matrix. Change of vectors and tensors by the following transformation,

$$\mathbf{k} = \mathrm{T}\mathbf{k}_{\text{new}}, \quad \mathbf{E} = \mathrm{T}\mathbf{E}_{\text{new}}, \quad \mathbf{H} = \mathrm{T}\mathbf{H}_{\text{new}}, \quad \hat{\varepsilon}_{\text{new}} = \widetilde{\mathrm{T}}\hat{\varepsilon}\mathrm{T}/\mathrm{Det}(\mathrm{T}), \;, \quad \hat{\mu}_{\text{new}} = \widetilde{\mathrm{T}}\hat{\mu}\mathrm{T}/\mathrm{Det}(\mathrm{T}), \tag{4}$$

leaves the system (2) the same in new variables. Then the choice of matrix T in the form

$$\hat{\mathrm{T}} = (\hat{\mu})^{-1/2}[\mathrm{Det}(\hat{\mu})]^{1/2} \tag{5}$$

allows to get (in Gaussian units)

$$\hat{\mu}_{\text{new}} = \hat{1}, \quad \hat{\varepsilon}_{\text{new}} = (\hat{\mu})^{-1/2}\hat{\varepsilon}(\hat{\mu})^{-1/2}. \tag{6}$$

It means that by a linear transformation of the coordinates in the k-space (but generally non-orthogonal, *i.e.* generally changing the angles) we were able to get rid of the magnetic anisotropy, and to reduce the problem to that with a purely dielectric anisotropy. Alternatively, similar trick allows to get rid of dielectric anisotropy, and to reduce the problem to that with a purely magnetic anisotropy. There are several interesting consequences of the above statements about the reduction possibilities.

First of all, the resulting new anisotropic tensor, *e.g.* of $\hat{\varepsilon}_{\text{new}}$ from eq. (6) is a symmetric tensor, and the corresponding surface, satisfying the equation for the length $|\mathbf{k}|$, has topological properties of pure dielectric case; it allows for not more than two axes of conical refraction, just as in the standard optical (dielectric-anisotropic) case. Second consequence is for an example of a rather artificial medium, where $\varepsilon_{ij} = \text{const} \cdot \mu_{ij}$, *i.e.* where some kind of an "impedance tensor" is proportional to unit tensor: $\hat{Z} = (\hat{\mu}/\hat{\varepsilon})^{1/2} = \hat{1}/\sqrt{\text{const}}$. If the dielectric (and hence the magnetic) susceptibilities are anisotropic, then the medium is evidently anisotropic. We were able to check that the equation for the length $|\mathbf{k}|$ describes in this case an ellipsoid: in coordinate system, where both $\hat{\varepsilon}$ and $\hat{\mu}$ are diagonal, that is where $P = S = R = 0$, $p = s = r = 0$, surface in $\mathbf{k}$-space becomes

$$\frac{k_x^2}{(B\cdot c=b\cdot C)} + \frac{k_y^2}{(A\cdot c=a\cdot C)} + \frac{k_z^2}{(A\cdot b=a\cdot B)} = \frac{\omega^2}{c_{\text{vac}}^2}. \tag{7}$$

What is truly remarkable, the two possible polarizations turn out to be degenerate, *i.e.* have identical lengths $|\mathbf{k}|$ for the given direction of $\mathbf{k}$. As a consequence, the propagation of a wave in such an anisotropic medium is not accompanied by any change of polarization – either by the birefringence, or by its rotation (by circular birefringence.)

Absence of birefringence for that case may be explained by following two symmetries of Maxwell equations. 1) Suppose, certain set of functions $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{D}(\mathbf{r},t) = \hat{\varepsilon}\mathbf{E}(\mathbf{r},t)$, $\mathbf{H}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t) = \hat{\mu}\mathbf{H}(\mathbf{r},t)$ satisfies the system (1) of Maxwell equations and material relationships. Then another set, with substitution

$$\mathbf{E}(\mathbf{r},t) \to \mathbf{H}(\mathbf{r},t), \quad \mathbf{H}(\mathbf{r},t) \to -\mathbf{E}(\mathbf{r},t); \quad \mathbf{D}(\mathbf{r},t) \to \mathbf{B}(\mathbf{r},t), \quad \mathbf{B}(\mathbf{r},t) \to -\mathbf{D}(\mathbf{r},t), \quad \hat{\varepsilon} \leftrightarrows \hat{\mu}, \tag{8}$$

satisfies the same system of equations (1), but with interchange $\hat{\varepsilon} \to \hat{\mu}$, $\hat{\mu} \to \hat{\varepsilon}$.

2) Besides that, Maxwell equations, together with material relationships, eqs. (1), are not just linear. They are also covariant to separate re-scaling of intensive variables **E** and **B**, connected by first vectored set of Maxwell equations. They are also invariant to separate re-scaling of extensive variables **D** and **H**, connected by the second vectored set of Maxwell equations. However, one has to perform simultaneous re-scaling of tensors of $\hat{\varepsilon}$ and $\hat{\mu}$:

$$\mathbf{E} = p\mathbf{E}_{\text{new}}, \quad \mathbf{B} = p\mathbf{B}_{\text{new}}, \quad \mathbf{D} = g\mathbf{D}_{\text{new}}, \quad \mathbf{H} = g\mathbf{H}_{\text{new}}, \quad \hat{\varepsilon} = (g/p)\hat{\varepsilon}_{\text{new}}, \quad \hat{\mu} = (p/g)\hat{\mu}_{\text{new}}. \tag{9}$$

Here p and g are some real constants. So, if all the components of tensor $\hat{\varepsilon}$ are proportional to corresponding components of tensor $\hat{\mu}$, then interchange $\hat{\varepsilon} \leftrightarrows \hat{\mu}$ in (8) may be considered just as re-scaling transformation from (9). It means that "**E**"- polarization has the same phase velocity as "**H**"- polarization, *i.e.* no birefringence. Q.E.D.

In this sense one may say, that the birefringence is not a "kinematic" effect of phase velocity of propagation, which is anisotropic here, since wave-vector surface is ellipsoid, eq. (7). On the contrary, birefringence is governed by the impedance, which (rather artificially) was made isotropic in this particular example. Special connection of impedance with polarization effects was also elucidated in [6-9].

To conclude, we have derived the analog of Fresnel equation for the case of the medium, where dielectric susceptibility and magnetic permeability tensors are both different from vacuum ones and are generally anisotropic. We have shown, that anisotropy of propagation is not a sufficient source of birefringence, while anisotropy of impedance tensor is.

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