# Impedance and canonical variables for the oscillators with "magnetic-type" forces.

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**Abstract**.

Canonical transformations are found for a system with the most general bi-linear Lagrangian or, equivalently, with the most general bi-linear Hamiltonian. Full account of possible "magnetic-type" forces is given. These canonical variables allow for the quantization of the oscillatory system in question.

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1. **Introduction**

An example of the parametric resonance is a swing. There is a consensus that if one changes the length $L(t)$ from the pivot axis to the mass center of the swinger, one gets efficient way to excite the oscillations. Folklore tradition is to attribute the parametric excitation of the swing to the modulation of frequency, $\omega(t) = \sqrt{g/L(t)}$, with $g$ [m/s$^2$], the gravity acceleration, and $L(t)$ [m], the length from the mass to the pivot axis. And the most appealing to the connoisseurs is the Mathieu equation, which we write here in the form

$$\frac{d^2x}{dt^2} + \omega_0^2\left[1 + \frac{2\omega_1}{\omega_0}\cos(pt)\right]x(t) = 0 \qquad (1)$$

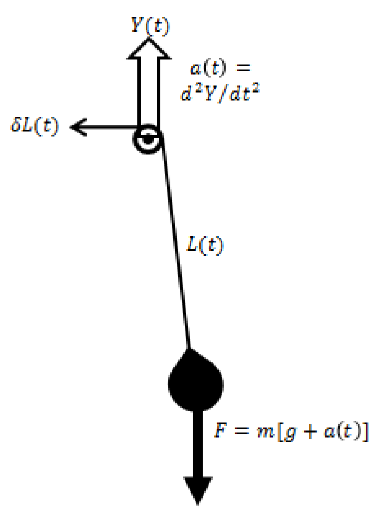


Figure. A pendulum, where the pivot point has its height $Y(t)$ modulated, as well as the length $L(t)$ from the mass to the pivot point. Vertical acceleration of the pivot point, $a(t) = d^2Y/dt^2$, results in effective change of gravity force, $mg_{eff}(t) = m[g_0 + a(t)]$.

If the "instantaneous frequency $\omega(t) \approx \omega_0 + \omega_1\cos(pt)$" is modulated with the period $T = 2\pi/p$, which is close enough to half of the period $T_0 = 2\pi/\omega_0$ of the un-modulated motion, then parametric instability of the solutions of the Mathieu equation (1) occurs. However, in a laboratory one can separately modulate the length $L(t)$, i.e. the distance from the axis to the mass, and effective gravity acceleration $g(t)$. The latter can be done by moving the pivot axis up and down with acceleration $a(t)$, so that $g_{eff}(t) = g_0 + a(t)$, see Figure. In the article [1] we tried to answer the following question.

**Is it really the modulation of instantaneous frequency $\omega(t) = \sqrt{g(t)/L(t)}$, that leads to parametric excitation ?**

Surprisingly, most texts, which discuss parametric processes, give the answer "YES" to this question, see e.g. [2-6], while the correct one should be "NO"! Indeed, we have introduced in [1] the notion of **impedance for the lumped systems**, including mechanical ones, and have shown that it is the modulation of pendulum's impedance, $Z(t) = \sqrt{m^2[L(t)]^3 \cdot g_{\mathrm{eff}}(t)}$, that results in parametric excitation. In other words, if the instantaneous frequency $\omega(t) = \sqrt{g(t)/L(t)}$ does not change in time, but the impedance $Z(t)$ does, then parametric excitation is possible. On the contrary, if the impedance $Z(t)$ is constant in time, but the instantaneous frequency $\omega(t) = \sqrt{g(t)/L(t)}$ is time-modulated, then there is absolutely no parametric excitation of the oscillator.

Contradiction with the well-known mathematical facts about the Mathieu equation (1) is resolved as follows. Namely, one should start with the system of two first-order Ordinary Differential Equations (ODEs). It is in the process of reduction of that system to the second order ODE eq. (1), where additional assumptions (sometimes correct, and sometimes incorrect) are usually made.

This situation has its analogs in electric *LC*-circuits with inductance *L* and capacitance *C*, where the frequency equals to $\omega = 1/\sqrt{LC}$, while the impedance is $Z = \sqrt{L/C}$. And again, most textbooks claim (incorrectly) that the modulation of frequency is the reason of parametric excitation, while actually it is only the modulation of impedance, which leads to parametric resonance.

## 2. Basic definitions and equations for Lagrangian, momenta, Hamiltonian

Consider a system of linear Ordinary Differential Equations (ODE), namely the one that may be produced as Euler-Lagrange equations of the variational principle with bi-linear Lagrangian. The corresponding Hamiltonian turns out to be bi-linear as well.

We will use the "vectors" and "transpose vectors" of *n* coordinates and corresponding velocities:

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \dots \\ x_n(t) \end{pmatrix}, \quad \dot{\mathbf{x}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dots \\ \dot{x}_n(t) \end{pmatrix}, \quad \mathbf{x}^{\mathrm{T}}(t) = \left(x_1(t), \dots x_n(t)\right), \quad \dot{\mathbf{x}}^{\mathrm{T}}(t) = \left(\dot{x}_1(t), \dots \dot{x}_n(t)\right),$$

We adopt a general Lagrange function to be bi-linear with respect to coordinates and velocities:

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = -\frac{1}{2}\mathbf{x}^{\mathrm{T}}\hat{\mathrm{K}}\mathbf{x} + \frac{1}{2}\dot{\mathbf{x}}^{\mathrm{T}}\hat{\mathrm{M}}\dot{\mathbf{x}} + \mathbf{x}^{\mathrm{T}}\hat{\beta}\dot{\mathbf{x}} \equiv -\frac{1}{2}x_i K_{ij} x_j + \frac{1}{2}\dot{x}_i M_{ij}\dot{x}_j + x_i \beta_{ij}\dot{x}_j \tag{2}$$

Here $K_{ij}(t)$, $M_{ij}(t)$, and $\beta_{ij}(t)$ are $n$ by $n$ matrices, $\hat{\mathrm{K}}$ and $\hat{\mathrm{M}}$ being symmetric ones, while generally $\hat{\beta}$ may be symmetric, and may be not; all three matrices may be time-dependent. As usual, the summation over the repeating indexes is assumed, and hat-symbol denotes matrix. The choice of signs and of the letters K, M and $\beta$ is aimed to remind elasticity constant K of an oscillator, mass M, and magnetic field B. The time-dependence of the antisymmetric part of $\hat{\beta}$ leads, according to the Faraday's Law of Electromagnetic Induction, to the curly electrical field, i.e. to the electromotive force.

Standard notations for the vectors of momenta **p** and of forces **f** yield

$$p_i = p_i(t) \equiv \frac{\partial L}{\partial \dot{x}_i} = M_{ij}\dot{x}_j + (\hat{\beta}^T)_{ij}x_j, \quad f_i = f_i(t) \equiv \frac{\partial L}{\partial x_i} = -K_{ij}x_j + (\hat{\beta})_{ij}\dot{x}_j, \qquad (3)$$
$$\mathbf{p} = \hat{\mathrm{M}}\dot{\mathbf{x}} + \hat{\beta}^{\mathrm{T}}\mathbf{x}, \quad \mathbf{f} = \hat{\beta}\dot{\mathbf{x}} - \hat{\mathrm{K}}\mathbf{x}.$$

One can express velocities via momenta:

$$\dot{\mathbf{x}} = \hat{\mathrm{M}}^{-1}\left(\mathbf{p} - \hat{\beta}^{\mathrm{T}}\mathbf{x}\right).$$

One needs this expression also for the transition to the Hamiltonian:

$$H(\mathbf{p},\mathbf{x},t) = \quad \mathbf{p}^{\mathrm{T}} \cdot \dot{\mathbf{x}}(\mathbf{p}) - L[\mathbf{x},\dot{\mathbf{x}}(\mathbf{p}),t] \quad = \quad \frac{1}{2}\left(\mathbf{p}^{\mathrm{T}} - \mathbf{x}^{\mathrm{T}}\hat{\beta}\right)\hat{\mathrm{M}}^{-1}\left(\mathbf{p} - \hat{\beta}^{\mathrm{T}}\mathbf{x}\right) + \frac{1}{2}\mathbf{x}^{\mathrm{T}}\hat{\mathrm{K}}\mathbf{x}. \qquad (4)$$

Standard Euler-Lagrange variational equations d**p**/dt = **f** are equivalent to canonical Hamilton's equations:

$$\dot{\mathbf{p}} = -\partial H / \partial \mathbf{x}^{\mathrm{T}}, \quad \dot{\mathbf{x}} = \partial H / \partial \mathbf{p}^{\mathrm{T}}, \qquad (5)$$

For our Hamiltonian (4) or, equivalently, for our Lagrangian (12) these equations are

$$\dot{p}_i(t) = \beta_{ij}\dot{x}_j - K_{ij}x_j \equiv (\hat{\beta}\hat{\mathrm{M}}^{-1})_{ij}\,p_j - (K + \hat{\beta}\hat{\mathrm{M}}^{-1}\hat{\beta}^T)_{ij}\,x_j, \quad \dot{x}_i(t) = (\mathrm{M}^{-1})_{ij}\,p_j - (\hat{\mathrm{M}}^{-1}\hat{\beta}^T)_{ij}\,x_j. \quad (6)$$

The same pair of equations may be written as

$$\frac{d}{dt}\begin{pmatrix}\mathbf{p}(t)\\ \mathbf{x}(t)\end{pmatrix} = V_{2n}\begin{pmatrix}\mathbf{p}(t)\\ \mathbf{x}(t)\end{pmatrix}, \quad V_{2n} = \begin{bmatrix}\hat{\beta}\hat{\mathrm{M}}^{-1} & -(\hat{\mathrm{K}} + \hat{\beta}\hat{\mathrm{M}}^{-1}\hat{\beta}^T)\\ \hat{\mathrm{M}}^{-1} & -\hat{\mathrm{M}}^{-1}\hat{\beta}^T\end{bmatrix}, \qquad (7)$$

with $V_{2n}$ being ($2n$ by $2n$) matrix. If all three matrices $\hat{\mathrm{M}}$, $\hat{\mathrm{K}}$ and $\hat{\beta}$ are time-independent, then one can also write these equations in the form of "Second Newton's Law":

$$\hat{\mathrm{M}}\frac{d^2\mathbf{x}}{dt^2} = (\hat{\beta} - \hat{\beta}^T)\frac{d\mathbf{x}}{dt} - \hat{\mathrm{K}}\mathbf{x}, \qquad \text{stationary case.} \qquad (8)$$

It means that in the stationary case the only important part of $\hat{\beta}$ -matrix is the skew-symmetric one, in accord with the expression **B** = curl(**A**(**r**)) in the familiar case of the motion of charged particle in the presence of magnetic field **B** in 3-dimensional space. By themselves, the equations (8) are valid for stationary case in any number of dimensions. Below in the Sections 3 through 6 we will work with the system (7) for momenta and coordinates for any number of dimensions.

For the simplest one-dimensional case Hamiltonian becomes

$$H(p,x,t) = \frac{p^2}{2m(t)} + \frac{K(t)x^2}{2},$$

for which the equations are

$$\frac{dp}{dt} = -K(t)x, \quad \frac{dx}{dt} = \frac{1}{m(t)}p, \tag{9}$$

One should introduce two different quantities: instantaneous frequency ω($t$) and instantaneous value of impedance[1] $Z(t)$ by the definitions

$$\omega(t) = \sqrt{\frac{K(t)}{m(t)}}, \quad Z(t) = \sqrt{K(t)m(t)}, \quad K(t) = \omega Z, \quad \frac{1}{m(t)} = \frac{\omega}{Z}. \tag{10}$$

Using frequency and impedance, one may re-write the Eqs. (9) as

$$\frac{dp}{dt} = -\omega(t)Z(t)\cdot x(t), \quad \frac{dx}{dt} = \frac{\omega(t)}{Z(t)}\cdot p(t), \tag{11}$$

A possible reason to call the quantity $Z(t)$ as "impedance" may be the following. If one adds the term of viscous force $f_{\text{visc}} = -Z_{\text{visc}}\,(dx/dt)$, then the dimensions of $Z_{\text{visc}}$ is the same as dimensions of our impedance $Z(t)$. Critical value of the damping corresponds to $Z_{\text{visc}} = 2Z$, when both eigenvalues for time evolution exponent switch from damped-oscillatory type to purely damped type. So-called quality factor $Q$ of an oscillatory system (e.g. of LRC-circuit in electronics) is

[1] Author is grateful to Prof. G.I. Barenblatt for pointing the following. Existing terminology in Mechanics uses word "Impedance" (according to Wikipedia) for frequency-dependent "*measure of how much a structure resists motion when subjected to a given force. It relates forces with velocities acting on a mechanical system. The mechanical impedance of a point on a structure is the ratio of the force applied at a point to the resulting velocity at that point.*" Author have used in [1] different meaning of the word "impedance", with the definition explained there at a length. Author is open to suggestions of a different term, but energetically insist on the importance of the quantity $Z = \sqrt{K \cdot m}$.

defined as $Q = (Z/Z_{visc})$; it equals to the ratio $Q = \omega_0/\gamma$ of the frequency $\omega_0$ to the damping constant $\gamma$ [1/second, for energy], $\gamma = Z_{visc}/m$. By the way, matrix $\beta(t)$ has also the dimensions of impedance. The dimension of impedance depends on the units of the coordinate $x$. Another physical meaning of the impedance in one-dimensional case is <u>aspect ratio of phase space cell</u> (see [1]), i.e. aspect ratio $\Delta p$ / $\Delta x$ for stationary elliptic trajectory.

One can perform one more canonical (symplectic) transformation to new variables: one new coordinate $X$ and one new momentum $P$:

$$p(t) = \sqrt{Z(t)} \cdot P(t), \quad x(t) = \frac{1}{\sqrt{Z(t)}} \cdot X(t). \tag{12}$$

Corresponding system of ODE for these new variables $X$ and $P$ is

$$\frac{d}{dt}\begin{bmatrix} P(t) \\ X(t) \end{bmatrix} = \begin{bmatrix} -g(t) & -\omega(t) \\ \omega(t) & g(t) \end{bmatrix}\begin{bmatrix} P(t) \\ X(t) \end{bmatrix}, \qquad g(t) = \frac{1}{2}\frac{d}{dt}\ln[Z(t)]. \tag{13}$$

Hamiltonian in this (generally non-stationary) case is

$$H_{\text{new}}(P, X, t) = g(t)PX + \frac{\omega(t)}{2}\left(X^2 + P^2\right), \tag{14}$$

with equations (13) being direct consequence of the Hamiltonian (14). It is convenient to introduce complex amplitudes $a(t)$, $a^*(t)$ and corresponding slow-varying complex amplitudes $c(t)$, $c^*(t)$ by the definitions

$$a(t) = \left[X(t) + iP(t)\right]\big/\sqrt{2\hbar}, \quad a^*(t) = \left[X(t) - iP(t)\right]\big/\sqrt{2\hbar}, \tag{15}$$

$$c(t) = a(t)\exp\left[i\int_0^t \omega(t')dt'\right], \quad c^*(t) = a^*(t)\exp\left[-i\int_0^t \omega(t')dt'\right]. \tag{16}$$

From the point of view of Classical Mechanics, the constant $2\hbar$ may be arbitrary, e.g. just 2. The convenience of interpreting $\hbar$ as Planck's constant can be seen in Quantum Mechanics, since the energy (Hamiltonian) for static $\omega$ and $Z$ is $H = \hbar\omega(aa^* + a^*a)/2$, with $a$ being interpreted as annihilation operator of one quantum $\hbar\omega$. Still within the Classical Mechanics, the <u>exact</u> linear ODE for "fast" amplitudes $a(t)$, $a^*(t)$ and for "slow varying" amplitudes $c(t)$, $c^*(t)$ become

$$da/dt = -i\omega a(t) + g(t)a^*(t), \;\; da^*/dt = g(t)a(t) + i\omega a^*(t), \tag{17}$$

$$g(t) = \frac{1}{2}\frac{d}{dt}\ln[Z(t)], \tag{18}$$

$$\frac{dc}{dt} = g(t)c^*(t)\exp\left[2i\int_0^t \omega(t')dt'\right], \qquad \frac{dc^*}{dt} = g(t)c(t)\exp\left[-2i\int_0^t \omega(t')dt'\right]. \tag{19}$$

These systems (17) or (19) allowed to come in [1] to an important conclusion. Namely, even in the case of time-dependent frequency $\omega(t)$, adiabatic invariant $aa^* \equiv cc^* = H/\hbar\omega$ is strictly conserved, if the impedance $Z(t)$ is constant in time and thus $g(t) \equiv 0$. Parametric excitation is completely absent in case of time-independent impedance. Moreover, if length $L(t)$ changes periodically (with frequency $2\omega_0$), but not for effective gravity acceleration $g_0$, then the use of Mathieu equation yields 3 times smaller exponent of parametric growth in comp comparison with the correct one.

### 3. Canonical transformation in multi-dimensional case with "magnetic" forces

Let us introduce now symmetric $n$-by-$n$ matrix $\hat{Z}$, generally complex-valued, which we call Z-pedance, to distinguish it from real matrix of impedance, which was previously introduced in [1] for the case without "magnetic-type forces", the latter forces being proportional to β. To do this, we define new complex vectors $\mathbf{b}(t)$ and $\mathbf{b^*}(t)$ via yet un-known matrix $\hat{Z}$ by

$$\mathbf{b}(t) = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\hat{Z}}\mathbf{x} + \frac{i}{\sqrt{\hat{Z}}}\mathbf{p}\right), \qquad \mathbf{b}^*(t) = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\hat{Z}^*}\mathbf{x} - \frac{i}{\sqrt{\hat{Z}^*}}\mathbf{p}\right) \tag{20}$$

Then it follows from Eqs. (20) that real vectors $\mathbf{p}(t)$ and $\mathbf{x}(t)$ are equal to

$$\mathbf{p}(t) = 2\left[\hat{Z}^{-1} + (\hat{Z}^*)^{-1}\right]^{-1}\sqrt{\frac{\hbar}{2}}\left(-i\frac{1}{\sqrt{\hat{Z}}}\mathbf{b} + i\frac{1}{\sqrt{\hat{Z}^*}}\mathbf{b}^*\right), \quad \mathbf{x}(t) = 2\left[\hat{Z} + \hat{Z}^*\right]^{-1}\sqrt{\frac{\hbar}{2}}\left(\sqrt{\hat{Z}}\mathbf{b} + \sqrt{\hat{Z}^*}\mathbf{b}^*\right) \tag{21}$$

Handling of the inverse of a sum of inverse matrices is facilitated by the identities eq. (40) below. One may show (as a result of straightforward, but tedious calculations for the time-independent matrices $\hat{M}$, $\hat{\beta}$, $\hat{K}$ and $\hat{Z}$) that the equations of motion (7), (8) for stationary system in terms of vectors $\mathbf{b}(t)$, $\mathbf{b}^*(t)$ become

$$\frac{d\mathbf{b}}{dt} = -i\hat{\omega}\mathbf{b} + i\hat{\rho}\mathbf{b}^*, \qquad \hat{\omega} = \hat{F}\cdot\hat{J}, \tag{22}$$

$$(\hat{J}^*)^T \equiv \hat{J} = 2\sqrt{\hat{Z}^*}\left(\hat{Z} + \hat{Z}^*\right)^{-1}\sqrt{\hat{Z}} \equiv 2\frac{1}{\sqrt{\hat{Z}^*}}\left(\hat{Z}^{-1} + (\hat{Z}^*)^{-1}\right)^{-1}\frac{1}{\sqrt{\hat{Z}}}, \tag{23}$$

$$\hat{F} = \frac{1}{2}\left(\sqrt{\hat{Z}}\hat{M}^{-1}\sqrt{\hat{Z}^*} + \frac{i}{\sqrt{\hat{Z}}}\hat{\beta}\hat{M}^{-1}\sqrt{\hat{Z}^*} - i\sqrt{\hat{Z}}\hat{M}^{-1}\hat{\beta}^T\frac{1}{\sqrt{\hat{Z}^*}} + \frac{1}{\sqrt{\hat{Z}}}\left(\hat{K} + \hat{\beta}\hat{M}^{-1}\hat{\beta}^T\right)\frac{1}{\sqrt{\hat{Z}^*}}\right), \tag{24}$$

$$\hat{\rho} = i\frac{1}{\sqrt{\hat{Z}}}\left[\hat{Z}\hat{M}^{-1}\hat{Z} + i\hat{\beta}\hat{M}^{-1}\hat{Z} + i\hat{Z}\hat{M}^{-1}\hat{\beta}^T - \left(\hat{K} + \hat{\beta}\hat{M}^{-1}\hat{\beta}^T\right)\right]\cdot\left(\hat{Z} + \hat{Z}^*\right)^{-1}\sqrt{\hat{Z}^*}. \qquad (25)$$

and equation, which is complex conjugate of (22), for $d\mathbf{b}^*(t)/dt$.

It is wroth noting that matrix $\hat{J}$ is Hermitian. Besides that, in the case of real matrix $\hat{Z}$, matrix $\hat{J}$ becomes unit matrix. With the aim of making $\mathbf{b}(t)$ to become an analog of annihilation vector-operator (up to a scale transformation) of a stationary quantum multi-dimensional oscillator, and making $\mathbf{b}^*(t)$ analog of creation vector-operator, we will require that those $\hat{\rho}$-terms in (22), which are proportional to $\mathbf{b}^*(t)$, vanish in stationary case. This requirement yields the equation, to which (generally complex) symmetric $n$-by-$n$ matrix $\hat{Z}$ of Z-pedance must satisfy:

$$\hat{Z}\hat{M}^{-1}\hat{Z} + i\hat{\beta}\hat{M}^{-1}\hat{Z} + i\hat{Z}\hat{M}^{-1}\hat{\beta}^T - \left(\hat{K} + \hat{\beta}\hat{M}^{-1}\hat{\beta}^T\right) = 0\,. \qquad (24)$$

<u>This equation constitutes one of the main results of the present work</u>. While by itself it is the equation from pure Classical Mechanics and does not contain Planck's constant $\hbar$, it may also be derived from the requirement that Gaussian function is an eigenfunction of the ground state of Quantum-Mechanical Hamiltonian.

It is convenient to split the stationary matrix $\hat{\beta}$ into symmetric and antisymmetric parts:

$$\hat{\beta} = \hat{\beta}_{\text{sym}} + \hat{\beta}_{\text{a}};\quad (\hat{\beta}_{\text{sym}})^T = \hat{\beta}_{\text{sym}};\quad (\hat{\beta}_{\text{a}})^T = -\hat{\beta}_{\text{a}}\,. \qquad (25)$$

Then that symmetric part may be eliminated: $\hat{Z} = \hat{Y} - i\hat{\beta}_{\text{sym}}$, so that eq. (24) becomes

$$\hat{Z} = \hat{Y} - i\hat{\beta}_{\text{sym}};\qquad \hat{Y}\hat{M}^{-1}\hat{Y} + i\hat{\beta}_{\text{a}}\hat{M}^{-1}\hat{Y} - i\hat{Y}\hat{M}^{-1}\hat{\beta}_{\text{a}} - \left(\hat{K} - \hat{\beta}_{\text{a}}\hat{M}^{-1}\hat{\beta}_{\text{a}}\right) = 0. \qquad (26)$$

In the case of absent β-type terms, the equation (24) (or (26)) has real solution,

$$\hat{Z}\hat{M}^{-1}\hat{Z} = \hat{K} \qquad \Rightarrow \qquad \hat{\mathrm{Z}} = \hat{\mathrm{M}}^{1/2}\left(\hat{\mathrm{M}}^{-1/2}\hat{\mathrm{K}}\hat{\mathrm{M}}^{-1/2}\right)^{1/2}\hat{\mathrm{M}}^{1/2} \equiv \hat{\mathrm{K}}^{1/2}\left(\hat{\mathrm{K}}^{1/2}\hat{\mathrm{M}}^{-1}\hat{\mathrm{K}}^{1/2}\right)^{-1/2}\hat{\mathrm{K}}^{1/2}, \qquad (27)$$

found and discussed earlier in [1].

It is well known, that the symmetric part of $\hat{\beta}$-matrix does not influence the trajectory in coordinate space in stationary case, see also equations (8) above. Equation (24) also possesses an explicit solution (which happens to be real) for special case: when, simultaneously, 1) for some or other reason, $\hat{\beta}$-matrix is antisymmetric (skew-symmetric in English), 2) $\hat{M} = m\cdot\hat{1}$ and 3)

$\hat{K} + \hat{\beta}\hat{M}^{-1}\hat{\beta}^T = q \cdot \hat{1}$. In that very special case Z-pedance matrix is real, proportional to unit matrix $\hat{1}$, and hence commutes with any antisymmetric matrix $\hat{\beta} = -\hat{\beta}^T$, so that

$$\hat{K} - \hat{\beta}_{\mathrm{a}}\hat{M}^{-1}\hat{\beta}_{\mathrm{a}} = K_1 \cdot \hat{1}, \quad \hat{M} = m \cdot \hat{1}, \quad \hat{\beta}_{\mathrm{a}} = -(\hat{\beta}_{\mathrm{a}})^T, \quad \Rightarrow \quad \hat{Z} = Z \cdot \hat{1}, \quad Z = \sqrt{K_1 \cdot m}, \qquad (28)$$

yielding standard one-dimensional expression $Z = \sqrt{K_1 \cdot m}$ of the impedance of system with lumped elements. The case of isotropic oscillator ($p_x$, $p_y$, $x$, $y$) in ($x$, $y$)-plane in the presence of $z$-component of magnetic field considered in [1], is described just by the equation (28).

Unfortunately for the physical interpretation of equation $d\mathbf{b}/dt = -i\hat{\omega}\mathbf{b}$, the matrix $\hat{\omega} = \hat{F} \cdot \hat{J}$ is not Hermitian, albeit is has the same eigenvalues as the actual frequency matrix $\hat{\Omega}$ of our system has, see below. This means that the transformations (20), (21) and their inverse, (2.3), (2.4), while succeeded in separating of positive and negative frequencies (under the condition of $\hat{\rho} = 0$, or Eq. (24)), failed to produce actual analogs of annihilation and creation amplitudes. In other words, imaginary and real parts, $\mathrm{Im}[\mathbf{b}(t)]$, $\mathrm{Re}[\mathbf{b}(t)]$

$$\mathrm{Im}[\mathbf{b}(t)] = (-i\mathbf{b} + i\mathbf{b}^*)/2, \qquad \mathrm{Re}[\mathbf{b}(t)] = (\mathbf{b} + \mathbf{b}^*)/2, \qquad (29)$$

do not constitute canonical variables (contrary to the beneficial case, when magnetic forces were absent, $\hat{\beta} = 0$, and thus when matrix $\hat{Z}$ was real, and when $\mathrm{Im}[\mathbf{b}(t)]$, and $\mathrm{Re}[\mathbf{b}(t)]$ were, up to constant factors, canonical variables, see [1].)

Going back to the general case, when Z-pedance matrix is symmetric, but may be complex-valued, it is convenient to define true (as we shall see) creation and annihilation amplitudes $\mathbf{a}(t) = \hat{D}\mathbf{b}(t)$ and $\mathbf{a}^*(t) = \hat{D}^*\mathbf{b}^*(t)$ as

$$\mathbf{a}(t) = \hat{D}\mathbf{b}(t) = \frac{1}{\sqrt{2\hbar}}\hat{D}\left(\sqrt{\hat{Z}}\mathbf{x} + \frac{i}{\sqrt{\hat{Z}}}\mathbf{p}\right), \quad \mathbf{a}^*(t) = \hat{D}^*\mathbf{b}^*(t) = \frac{1}{\sqrt{2\hbar}}\hat{D}^*\left(\sqrt{\hat{Z}^*}\mathbf{x} - \frac{i}{\sqrt{\hat{Z}^*}}\mathbf{p}\right). \quad (30)$$

Then it follows from eqs. (20) that

$$\mathbf{p}(t) = 2\left[\hat{Z}^{-1} + (\hat{Z}^*)^{-1}\right]^{-1}\sqrt{\frac{\hbar}{2}}\left(-i\frac{1}{\sqrt{\hat{Z}}}\hat{D}^{-1}\mathbf{a} + i\frac{1}{\sqrt{\hat{Z}^*}}(\hat{D}^*)^{-1}\mathbf{a}^*\right), \qquad (31)$$

$$\mathbf{x}(t) = 2\left[\hat{Z} + \hat{Z}^*\right]^{-1}\sqrt{\frac{\hbar}{2}}\left(\sqrt{\hat{Z}}\hat{D}^{-1}\mathbf{a} + \sqrt{\hat{Z}^*}(\hat{D}^*)^{-1}\mathbf{a}^*\right), \tag{32}$$

with yet unknown transformation ($n$-by-$n$) matrix $\hat{D}$. Now we shall choose the matrix $\hat{D}$ satisfying equation

$$\hat{D}\cdot(\hat{D}^*)^T = \hat{J}, \quad \hat{J} = 2\sqrt{\hat{Z}^*}\left(\hat{Z} + \hat{Z}^*\right)^{-1}\sqrt{\hat{Z}} \equiv 2\frac{1}{\sqrt{\hat{Z}^*}}\left(\hat{Z}^{-1} + (\hat{Z}^*)^{-1}\right)^{-1}\frac{1}{\sqrt{\hat{Z}}}, \quad \hat{J} = (\hat{J}^*)^T. \tag{33}$$

Since $\hat{J}$ is Hermitian matrix and, apparently, positive-definite, one can expect that $\hat{D}$ matrix is Hermitian as well, and hence $\hat{D}$ is just a square root of $\hat{J}$:

$$\hat{D} = (\hat{D}^*)^T; \quad \hat{D}\cdot\hat{D} = \hat{J} \quad \Rightarrow \quad \hat{D} = \sqrt{\hat{J}}. \tag{34}$$

Assuming that the solution of (34) is found, we can write the equation for complex vector **a**(t):

$$\frac{d\mathbf{a}}{dt} = -i\hat{\Omega}\mathbf{a}, \quad \hat{\Omega} = \hat{D}\hat{F}(\hat{D}^*)^T \equiv \hat{D}\hat{F}\hat{D},$$
$$\hat{F} = \frac{1}{2}\left[\sqrt{\hat{Z}}\hat{M}^{-1}\sqrt{\hat{Z}^*} + \frac{i}{\sqrt{\hat{Z}}}\hat{\beta}\hat{M}^{-1}\sqrt{\hat{Z}^*} - i\sqrt{\hat{Z}}\hat{M}^{-1}\hat{\beta}^T\frac{1}{\sqrt{\hat{Z}^*}} + \frac{1}{\sqrt{\hat{Z}}}\left(\hat{K} + \hat{\beta}\hat{M}^{-1}\hat{\beta}^T\right)\frac{1}{\sqrt{\hat{Z}^*}}\right]. \tag{35}$$

Matrix $\hat{\Omega}$ has the physical sense of angular frequency and has dimensions [1/second]. The Hermitian character of the matrices $\hat{F}$ and $\hat{\Omega}$ is evident from eqs. (35). This matrix $\hat{\Omega}$ may be reduced to the diagonal form (with real positive eigenvalues $\Omega^{(i)}$, $i$=1, 2, …, $n$) by another unitary transformation.

But even prior to diagonalization of $\hat{\Omega}$, one can introduce real new canonical momenta **P**($t$) and real new canonical coordinates **X**($t$) by the definitions

$$\mathbf{P}(t) = \sqrt{\frac{\hbar}{2}}\left(-i\mathbf{a} + i\mathbf{a}^*\right) = \hat{S}\mathbf{p} + \hat{U}\mathbf{x}, \quad \mathbf{X}(t) = \sqrt{\frac{\hbar}{2}}\left(\mathbf{a} + \mathbf{a}^*\right) = \hat{V}\mathbf{p} + \hat{W}\mathbf{x}. \tag{36}$$

Here $n$-by-$n$ matrices $\hat{S}$, $\hat{U}$, $\hat{V}$ and $\hat{W}$ do not contain any Planck's constant $\hbar$, they are

$$\hat{S} = \frac{1}{2}\left(\hat{D}\frac{1}{\sqrt{\hat{Z}}} + \hat{D}^*\frac{1}{\sqrt{\hat{Z}^*}}\right), \quad \hat{U} = \frac{1}{2i}\left(\hat{D}\sqrt{\hat{Z}} - \hat{D}^*\sqrt{\hat{Z}^*}\right),$$
$$\hat{V} = \frac{i}{2}\left(\hat{D}\frac{1}{\sqrt{\hat{Z}}} - \hat{D}^*\frac{1}{\sqrt{\hat{Z}^*}}\right), \quad \hat{W} = \frac{1}{2}\left(\hat{D}\sqrt{\hat{Z}} + \hat{D}^*\sqrt{\hat{Z}^*}\right); \tag{37}$$

they constitute the elements of 2$n$-by-2$n$ symplectic matrix $\hat{Z}_{2n}$,

$$\begin{pmatrix} \mathbf{P} \\ \mathbf{X} \end{pmatrix} = \hat{Z}_{2n} \begin{pmatrix} \mathbf{p} \\ \mathbf{x} \end{pmatrix}, \quad \hat{Z}_{2n} = \begin{pmatrix} \hat{S} & \hat{U} \\ \hat{V} & \hat{W} \end{pmatrix}. \tag{38}$$

Reminder about notations of the matrices $\hat{S}$, $\hat{U}$, $\hat{V}$ and $\hat{W}$ from the previous paper [1]: they are going in alphabetic order from left to right in the upper row of eq. (38), and then again from left to right in lower row; however, letter "*T*" is skipped, since one needs it for the notation of transposed matrix.

Symplecticity conditions for the linear transformation (38) have the following form:

$$\hat{S} \cdot \hat{U}^T = \left(\hat{S} \cdot \hat{U}^T\right)^T, \quad \hat{V} \cdot \hat{W}^T = \left(\hat{V} \cdot \hat{W}^T\right)^T, \quad \hat{S} \cdot \hat{W}^T - \hat{U} \cdot \hat{V}^T = \hat{1}. \tag{39}$$

(see e.g. [1].) Rather unpleasant and lengthy manipulations (with the use of relationship (33) and identities

$$\left(\frac{1}{\hat{A}} + \frac{1}{\hat{B}}\right)^{-1} \equiv \hat{B}\left(\hat{A} + \hat{B}\right)^{-1} \hat{A} \equiv \hat{A}\left(\hat{A} + \hat{B}\right)^{-1} \hat{B}, \tag{40}$$

which are valid in assumption of existence of all of those inverse matrices, allow to prove (see Appendix B), that the above definitions of $\hat{S}$, $\hat{U}$, $\hat{V}$ and $\hat{W}$ yield symplectic matrix $\hat{Z}_{2n}$.

## 4. Numerical example

As a test of the technique suggested, one can show the numerical solutions of 1) eq. (24), and 2) of eq. (33) for some particular stationary problem of 3-dimensional oscillator in a magnetic field. The anisotropy of that oscillator had rather general form, coupling *z*-motion with the motion in (*x*, *y*)-plane, the latter plane being perpendicular to magnetic field. For simplicity, the mass was considered to be unit scalar. The way eq. (24) was solved was iterative procedure: start with $\hat{Z}_{\text{old}}$ and find $Z_{\text{double new}}$ from

$$Z_{\text{new}} = \frac{1}{2}\left(\hat{K} - \hat{\beta}_{\text{a}}\hat{M}^{-1}\hat{\beta}_{\text{a}} - i\hat{\beta}_{\text{a}}\hat{M}^{-1}\hat{Z}_{\text{old}} + i\hat{Z}_{\text{old}}\hat{M}^{-1}\hat{\beta}_{\text{a}}\right)\frac{1}{\hat{Z}_{\text{old}}}\hat{M} + \frac{1}{2}\hat{Z}_{\text{old}}, \quad Z_{\text{double new}} = \frac{1}{2}\left[Z_{\text{new}} + (Z_{\text{new}})^T\right].$$

Here is the particular numerical example:

$$\hat{M} = \hat{1}, \quad \hat{\beta} = -\hat{\beta}^T = \hat{\beta}_{\text{a}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{K} = \begin{pmatrix} 1 & 0.3 & 0.2 \\ 0.3 & 1 & 0.5 \\ 0.2 & 0.5 & 0.8 \end{pmatrix}. \tag{41}$$

Eigenvalues of the matrix $\hat{K}-\hat{\beta}_{\mathrm{a}}\hat{M}^{-1}\hat{\beta}_{\mathrm{a}}$ are approximately 0.613, 1.733, 2.454. By iteration procedure the solution of the equation (24). This solution is

$$\hat{Z}=\begin{pmatrix} 1.442326-i\cdot 0.130965 & 0.19078+i\cdot 0.044901 & 0.08531-i\cdot 0.122509 \\ 0.19078+i\cdot 0.044901 & 1.348378+i\cdot 0.128776 & 0.274323+i\cdot 0.031261 \\ 0.08531-i\cdot 0.122509 & 0.274323+i\cdot 0.031261 & 0.856423+i\cdot 0.00219 \end{pmatrix}. \quad (42)$$

The error may be estimated by the right-hand side of the equation (24). That right-hand side turned to be the matrix with the modulus of each element smaller than $10^{-0.1\mathrm{Q}}$, where number of iterations was $N \approx 10^{Q}$. Hermitian matrix $\hat{D}=\sqrt{\hat{J}}$ and the expression for the corresponding Hermitian matrix $\hat{\Omega}$ from eq. (35) were also found. Eigenvalues of 3-by-3 matrix $\hat{\Omega}$ from were respectively 2.4279396, 0.3250763 and 0.89411108. Identical eigenvalues (plus a set of negative of these) were obtained by direct consideration of 3 second-order equations (8). The validity of the symplecticity relationships (39) was verified, and it was perfect !

**5. Quantum Mechanics of the system**

Besides its own classical value, the canonical transformation (36), (37) allows for quantization of the system with the most general bilinear Lagrangian (2) or Hamiltonian (4), with arbitrary anisotropy and with time-non-symmetric "magnetic" β-terms. In particular, wavefunction of the ground state in coordinate representation is multi-dimensional Gaussian:

$$\psi(\mathbf{x})=\mathrm{const}\cdot\exp\left(-\frac{1}{2\hbar}\sum_{i,k}Z_{ik}x_i x_k\right). \quad (43)$$

Indeed, direct (albeit somewhat tedious) calculation shows that the stationary Hamiltonian,

$$H(\mathbf{p},\mathbf{x})=\frac{1}{2}\left(\mathbf{p}^{\mathrm{T}}-\mathbf{x}^{\mathrm{T}}\hat{\beta}\right)\hat{\mathrm{M}}^{-1}\left(\mathbf{p}-\hat{\beta}^{\mathrm{T}}\mathbf{x}\right)+\frac{1}{2}\mathbf{x}^{\mathrm{T}}\hat{\mathrm{K}}\mathbf{x}, \qquad p_j=-i\hbar\frac{\partial}{\partial x_j}, \quad (44)$$

has the function (43) as an eigenfunction, if eq. (24) holds. By the way, one can make a hypothesis that the wavefunction has the form (43). Then the requirement that it is valid eigenstate of Hamiltonian (44) leads to the equation (24). For some readers this may be an easier way of derivation of the main "Classical" equation (24).

The fact that it is indeed the ground state, may be verified by applying the *n*-dimensional vector-operator of annihilation $\mathbf{a}$ from (36) to the wavefunction (43) with account of

$p_j = -i\hbar(\partial/\partial x_j)$. This application yields zero for each of $n$ components of that vector, as it should be for the ground state.

The expression (43) is valid in most general case with arbitrary (transpose-symmetric or not) $\hat{\beta}$-terms, when complex-valued symmetric $n$-by-$n$ matrix $\hat{Z}$ of Z - pedance satisfies the equation (24), established in this work. As for the expressions for operators of individual creation and annihilation operators of particular eigenmodes, one can use standard software packages of finding eigenvalues of matrix $\hat{\Omega}$ and eigenvectors as defined by eq. (35).

## 6. Ion in a magnetic trap

Consider now the particular example: the problem of motion of an ion confined in ($x$, $y$)-plane by homogeneous magnetic field $\mathbf{B} = \mathbf{e}_z B$. In the absence of any other forces, this field leads to the classical rotary motion, with arbitrary ($x$, $y$)-position of the center, arbitrary radius of the orbit and with cyclotron frequency

$$\mathbf{\Omega} = -\mathbf{e}_z \Omega_{\text{cyclotron}}, \quad \Omega_{\text{cyclotron}} = qB/m \equiv 2L; \quad L > 0. \tag{45}$$

Here $q$ is the charge of the ion, $m$, its mass, $L = \Omega_{\text{cyclotron}}/2$ is Larmor frequency for this ion and in this field. We adopt this expression for the vector-potential:

$$\mathbf{A}(x, y) = B[y\mathbf{e}_x - x\mathbf{e}_y]/2; \quad \mathbf{B} = curl[\mathbf{A}(x, y)] = B\mathbf{e}_z, \tag{46}$$

which was chosen for its axial symmetry. We add also the electrostatic potential energy in the form

$$U(x, y) = qD(x^2 + y^2)/2 \equiv m\omega_0^2(x^2 + y^2)/2 \equiv -m\kappa_0^2(x^2 + y^2)/2. \tag{47}$$

Here $D = (\omega_0)^2 m/q$, with $(\omega_0)^2 > 0$ for axially symmetric potential well, $\omega_0$ [rad/second] being the frequency of oscillations $\exp(\pm i\omega_0 t)$ in this well in the absence of magnetic field. Another possibility is that $\omega_0^2 \equiv -\kappa_0^2 < 0$; then this energy $U(x, y)$ corresponds to potential hill, with time-dependence of coordinates and momenta $\propto \exp(\pm\kappa_0 t)$ in the absence of magnetic field. Since electrostatic potential in vacuum satisfies Laplace equation, one has in 3D axially symmetric case:

$$U(x, y, x) = qD(x^2 + y^2 - 2z^2)/2. \tag{48}$$

If it is to provide stability (i.e. potential well) along $z$-coordinate, then it becomes potential hill in ($x$, $y$)-plane, $\omega_0^2 \equiv -\kappa_0^2 < 0$. In this case the purpose of magnetic field is to suppress instability, introduced by the potential hill.

Equations of motion for $(p_x, p_y, x, y)$ for this system are

$$\frac{d}{dt}\begin{pmatrix} p_x \\ p_y \\ x \\ y \end{pmatrix} = \hat{E}\begin{pmatrix} p_x \\ p_y \\ x \\ y \end{pmatrix}, \qquad \hat{E} = \begin{pmatrix} 0 & L & m(\kappa_0^2 - L^2) & 0 \\ -L & 0 & 0 & m(\kappa_0^2 - L^2) \\ 1/m & 0 & 0 & L \\ 0 & 1/m & -L & 0 \end{pmatrix}. \qquad (49)$$

Here $-\omega_0^2 \equiv \kappa_0^2 > 0$ for the most interesting case of 2D potential hill. Eigenvalues of the time-evolution matrix $\hat{E}$, i.e. the roots of its characteristic polynomial $P(\lambda) = \det[\hat{E} - \lambda \cdot \hat{1}]$, are

$$\lambda_{1,2,3,4} = \pm i \cdot \omega_{\alpha,\beta}, \quad \omega_\alpha = L + \sqrt{L^2 - \kappa_0^2}, \quad \omega_\beta = L - \sqrt{L^2 - \kappa_0^2}\,. \qquad (50)$$

If $\omega_0^2 \equiv -\kappa_0^2 > 0$, then the motion in (*x*, *y*)-plane is evidently stable in 2D potential well, additionally supported by the magnetic field. But even for $-\omega_0^2 \equiv \kappa_0^2 > 0$, i.e. for the 2D potential hill, the magnetic Lorentz force may provide the stability in (*x*, *y*)-plane, i.e. allows to keep the frequencies $\omega_{\alpha,\beta}$ real, if the steepness of hill is not too large: $|\kappa_0| < L$.

Solution of equation (24) for *Z*-pedance matrix in the latter “sable” case $|\kappa_0| < L$ is

$$\hat{Z} = \hat{1} \cdot Z; \qquad Z = m\sqrt{L^2 - \kappa_0^2} \equiv m\sqrt{L^2 + \omega_0^2}\,. \qquad (51)$$

It means that Gaussian function of the ground state (43) turns to be real in the “stable” case and for this particular choice of vector-potential (46), in spite of the presence of the time-symmetry violation by the magnetic field. However, the excited states with energies

$$E(n_\alpha, n_\beta) = \hbar\omega_\alpha \cdot (n_\alpha + \tfrac{1}{2}) + \hbar\omega_\beta \cdot (n_\beta + \tfrac{1}{2}) \qquad (52)$$

have complex-valued wave functions.

## 7. Conclusion

We have considered in this work the most general dynamic system with bi-linear Lagrangian or, what is equivalent, with bi-linear Hamiltonian. Both Classical Mechanics and

Quantum Mechanics of such a system are considered. We have found a possible way, how to perform the symplectic (canonical) transformation (36 – 38) to the new real momenta **P**($t$) and real coordinates **X**($t$), yielding complex classical amplitudes **a**($t$) and **a***($t$). The latter amplitudes are complete analogs of quantum annihilation and creation operators. Wave function in coordinate representation is found for the ground state of the system. All other states may be found by sequential applications of creation operator.

While the canonical (symplectic) character of the transformations found in this note is valid both for stationary and non-stationary systems, we did not discuss here the non-stationary behavior in the way we did it in [1] for non-magnetic systems.

**Appendix A**

In the process of using iterations for the solution of matrix equations (24) and (33), and for the use of other equations of this paper, we need the formula for the function $f(\hat{A})$ of an arbitrary $n$-by-$n$ matrix $\hat{A}$. Corresponding expression can be found in e.g. the book by F.R. Gantmacher [7]. Since that book is rather rare, we present in this Appendix the corresponding formula. It may be called "Lagrange interpolation formula", but it is actually an exact one, as a consequence of Hamilton-Cayley theorem. To be concrete, we present it for 3-by-3 matrix $\hat{A}$. Namely, for an arbitrary analytic function $f(z)$ of one complex argument $z$ one has

$$f(\hat{A}) = f(\lambda_1)\cdot\frac{(\hat{A}-\lambda_2\cdot\hat{1})\cdot(\hat{A}-\lambda_3\cdot\hat{1})}{(\lambda_1-\lambda_2)\cdot(\lambda_1-\lambda_3)} + f(\lambda_2)\cdot\frac{(\hat{A}-\lambda_1\cdot\hat{1})\cdot(\hat{A}-\lambda_3\cdot\hat{1})}{(\lambda_2-\lambda_1)\cdot(\lambda_2-\lambda_3)} + f(\lambda_3)\cdot\frac{(\hat{A}-\lambda_1\cdot\hat{1})\cdot(\hat{A}-\lambda_2\cdot\hat{1})}{(\lambda_3-\lambda_1)\cdot(\lambda_3-\lambda_2)}\ . \quad \text{(A.1)}$$

Here $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the three eigenvalues of the matrix $\hat{A}$, i.e. the roots of characteristic polynomial

$$P_A(\lambda) = \det[\hat{A}-\lambda\cdot\hat{1}]. \quad \text{(A2)}$$

Generalization of "Lagrange interpolation formula" (A.1) to any other size $n$ of $n$-by-$n$ matrix is evident. The Hamilton-Cayley theorem, used in derivation of (A.1), states in those notations, that

$$P_A(\hat{A}) = \hat{0}\,, \quad \text{(A.3)}$$

and hence $n$-th and all higher powers of arbitrary $n$-by-$n$ matrix $\hat{A}$ may be expressed through the lower powers of that matrix [ ($n$–1)st, ($n$–2)nd, etc, down to zero power of $\hat{A}$, i.e. to $\hat{1}$ ].

The use of formula of the type of (A.1), unfortunately, does require finding of the eigenvalues of our matrix $\hat{A}$, but, fortunately, does not require finding of its eigenvectors and their use. That use may be especially inconvenient, if the matrix $\hat{A}$ is not-Hermitian, and thus its eigenvectors are generally not mutually orthogonal.

## Appendix B

Here we present some formulae for the matrices, involved in the proofs of eq. (36)-(38). By themselves they are reasonably simple. However, their derivation takes certain patience. The Hermitian matrices $\hat{J}$ and $\hat{D} = \sqrt{\hat{J}}$ were defined by eqs. (23), (33) and (34). Those definitions allow derivation of formulae for real, symmetric and hence Hermitian matrices (B.1) and (B.2):

$$\left(\hat{Z} + \hat{Z}*\right) \equiv 2\sqrt{\hat{Z}}(\hat{J})^{-1}\sqrt{\hat{Z}*} \equiv 2\sqrt{\hat{Z}*}(\hat{J}*)^{-1}\sqrt{\hat{Z}} \quad , \tag{B.1}$$

$$\left(\hat{Z}^{-1} + (\hat{Z}*)^{-1}\right) \equiv 2\frac{1}{\sqrt{\hat{Z}}}(\hat{J})^{-1}\frac{1}{\sqrt{\hat{Z}*}} \equiv 2\frac{1}{\sqrt{\hat{Z}*}}(\hat{J}*)^{-1}\frac{1}{\sqrt{\hat{Z}}}. \tag{B.2}$$

Moreover, for the mechanically stable system with non-negative-definite Hamiltonian (1.5) the matrices (B.1) and (B.2) are themselves non-negative-definite.

Besides that, the following matrices $\hat{N}$ and $\hat{P}$ are also Hermitian, and

$$\hat{N} = (\hat{N}*)^T = \frac{1}{\sqrt{\hat{Z}}}\sqrt{\hat{Z}*} + \sqrt{\hat{Z}}\frac{1}{\sqrt{\hat{Z}*}} \equiv 2(\hat{J})^{-1} \tag{B.3}$$

$$\hat{P} = (\hat{P}*)^T = \hat{N}^T = \hat{N}* = \frac{1}{\sqrt{\hat{Z}*}}\sqrt{\hat{Z}} + \sqrt{\hat{Z}*}\frac{1}{\sqrt{\hat{Z}}} \equiv 2(\hat{J}*)^{-1} \tag{B.4}$$

$$\hat{D}\cdot\hat{N}\cdot\hat{D} \equiv \hat{D}^T\cdot\hat{N}^T\cdot\hat{D}^T = 2\cdot\hat{1} \tag{B.5}$$

$$\hat{D}^T\cdot\hat{P}\cdot\hat{D}^T = \hat{D}\cdot\hat{P}^T\cdot\hat{D} = 2\cdot\hat{1} \tag{B.6}$$

As a result,

$$\hat{S}\cdot\hat{U}^T - \hat{U}\cdot\hat{S}^T = \frac{1}{4i}\left\{-\hat{D}\cdot\hat{N}\cdot\hat{D} + \hat{D}^T\cdot\hat{P}\cdot\hat{D}^T\right\} = \hat{0}. \tag{B.7}$$

$$\hat{V}\cdot\hat{W}^T - \hat{W}\cdot\hat{V}^T = \frac{i}{4}\left\{\hat{D}\cdot\hat{N}\cdot\hat{D} - \hat{D}^T\cdot\hat{P}\cdot\hat{D}^T\right\} = \hat{0}. \tag{B.8}$$

$$\hat{S}\cdot\hat{W}^T - \hat{U}\cdot\hat{V}^T = \frac{1}{4}\left\{\hat{D}\cdot\hat{N}\cdot\hat{D} + \hat{D}^T\cdot\hat{P}\cdot\hat{D}^T\right\} = \hat{1}. \tag{B.9}$$

Thus the symplecticity conditions (see e.g. [1]) are satisfied.

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