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What is the maximum attainable visibility by a partially coherent electromagnetic field in Young's double-slit interference?

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Abstract: What is the maximum visibility attainable in double-slit interference by an electromagnetic field if arbitrary – but *reversible* – polarization *and* spatial transformations are applied? Previous attempts at answering this question for electromagnetic fields have emphasized maximizing the visibility under local polarization transformations. I provide a definitive answer in the general setting of partially coherent electromagnetic fields. An analytical formula is derived proving that the maximum visibility is determined by only the two smallest eigenvalues of the 4×4 two-point coherency matrix associated with the electromagnetic field. This answer reveals, for example, that any two points in a spatially *in*coherent *scalar* field can always achieve *full* interference visibility by applying an appropriate reversible transformation spanning both the polarization and spatial degrees of freedom – *without* loss of energy. Surprisingly, almost all current measures predict zero-visibility for such fields. This counter-intuitive result exploits the higher dimensionality of the Hilbert space associated with *vector* – rather than *scalar* – fields to enable coherency conversion between the field's degrees of freedom.

OCIS codes: (030.1640) Coherence; (260.0260) Physical optics; (260.3160) Interference.

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1. Introduction

Thomas Young's report on the observation of double-slit interference [1] marks a landmark in our understanding of the nature of light [2]. Double-slit interference is an essential methodology for evaluating the spatial coherence of optical fields and remains an important conceptual tool in both classical [3] and quantum [4–6] optics. Spatial coherence – exemplified by high-visibility double-slit interference – may nevertheless be obfuscated by polarization [7–10]. Indeed, the visibility can be modified even by *reversible* (unitary) polarization devices placed at the slits [11, 12], thereby reducing the operational value of interference visibility as a hallmark of coherence for electromagnetic (EM) fields.

A range of answers have been provided in the literature to the following question: what is the maximum visibility attainable by a partially coherent and partially polarized EM field in Young's double slit experiment? The multiplicity of answers to this question is natural because the constraints placed on the maximization procedure have varied. In general, however, investigations have emphasized *local polarization transformations* implemented at each point – whether unitary (reversible and energy-conserving) [11, 12] or otherwise [13–15]. Such a state of affairs is not satisfying because the spatial and polarization degrees of freedom (DoFs) are not treated on the same footing, and spatial transformations are not included in the analysis.

Here, I address the following question: what is the maximum visibility of double-slit interference that may be observed from two points in a partially coherent and partially polarized EM field if arbitrary unitary transformations ('unitaries' hereon for brevity) may be applied

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to *either* of its DoFs (spatial or polarization) or *jointly* to both? This is a larger family of transformations than has been considered to date. EM fields that may be unitarily inter-converted are members of an equivalency class that share the same unitary invariants, and studying the maximum visibility attainable under the most general spatial-polarization unitaries helps identify an intrinsic field-invariant that is independent of our manner of interrogation. This maximum visibility is shown to depend only on the smallest two eigenvalues of the 4×4 two-point vector coherency matrix of the EM field. Furthermore, I demonstrate that most previous measures of two-point visibility predict zero-visibility for a wide class of fields that may nevertheless exhibit high visibility once the class of unitaries encompassing joint spatial-polarization transformations is considered in lieu of only local polarization unitaries. In other words, by examining the full Hilbert space describing the polarization *and* spatial DoFs for EM fields and symmetrizing their treatment, a higher visibility can be attained. In answering the titular question, it is found that *scalar* fields lacking *any* spatial coherence can nevertheless exhibit full interference visibility by reversible conversion – without loss of energy – to an unpolarized but spatially coherent field. This process of 'coherency conversion' between the field DoFs can potentially be exploited in protecting a beam from the deleterious impact of a randomizing medium.

The paper is organized as follows. First, I briefly review the standard description of polarization and spatial coherence – each treated independently – via 2×2 coherency matrices to fix the notation, before introducing the 4×4 vector-field coherency matrix describing jointly polarization and spatial coherence, followed by defining the problem that is tackled in this paper. Young's double-slit interference is a venerable problem in optics, and I therefore briefly review in Section 3 previous relevant investigations of the maximal interference visibility of an EM field via local polarization unitaries to properly situate the new result developed here. In Section 4 I obtain a closed-form expression for the maximum visibility attainable when an EM field is subject to a general non-separable polarization-spatial unitary transformation. Examples that apply this new formula and compare it to the visibility predicted by previous analyses are presented in Section 5, in addition to a comparison with previous efforts that have considered maximizing the visibility under *non*-unitary transformations. Finally, I provide in Section 6 an example of 'coherency conversion' before presenting the conclusions.

2. Statement of the problem

2.1. 2×2-*Matrix description of partial polarization and partial spatial coherence*

Partial polarization at a position $\vec{r}$ in a field is described via a 2×2 Hermitian positive semi-definite *polarization* coherency matrix $\mathbf{G}_{\mathrm{p}}=\begin{pmatrix}G^{\mathrm{HH}} & G^{\mathrm{HV}}\\ G^{\mathrm{VH}} & G^{\mathrm{VV}}\end{pmatrix}$, where $G^{jj'}=\langle E_j(\vec{r})E^*_{j'}(\vec{r})\rangle$, $j,j'=$ H, V are the horizontal and vertical polarization components, respectively, and normalized such that $G^{\mathrm{HH}}+G^{\mathrm{VV}}=1$ [3, 16, 17]. The degree of polarization is defined as $D_{\mathrm{p}}=\lambda_{\mathrm{H}}-\lambda_{\mathrm{V}}$, where $\lambda_{\mathrm{H}}\geq\lambda_{\mathrm{V}}\geq0$ are the eigenvalues of $\mathbf{G}_{\mathrm{p}}$ [18] obtained by diagonalization via a polarization unitary.

Spatial coherence at two points $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$ in a *scalar* quasi-monochromatic field may be defined in a similar fashion via a 2×2 Hermitian positive semi-definite *spatial* coherency matrix $\mathbf{G}_{\mathrm{s}}=\begin{pmatrix}G_{\mathrm{aa}} & G_{\mathrm{ab}}\\ G_{\mathrm{ba}} & G_{\mathrm{bb}}\end{pmatrix}$, where $G_{kk'}=\langle E(\vec{r}_k)E^*(\vec{r}_{k'})\rangle$, $k,k'=\mathrm{a},\mathrm{b}$, and $G_{\mathrm{aa}}+G_{\mathrm{bb}}=1$. The double-slit interference visibility is $V=2|G_{\mathrm{ab}}|$. It was recognized early on by Zernike [19] that V so-defined is not a unitary invariant, but can in fact be changed upon applying spatial unitaries. In analogy to D_{p}, a unitarily invariant degree of spatial coherence is defined, $D_{\mathrm{s}}=\lambda_{\mathrm{a}}-\lambda_{\mathrm{b}}$, where $\lambda_{\mathrm{a}}\geq\lambda_{\mathrm{b}}\geq0$ are the eigenvalues of $\mathbf{G}_{\mathrm{s}}$. It is straightforward to show that

$$D_{\mathrm{s}}=\max\{G_{\mathrm{aa}}-G_{\mathrm{bb}}\}=V_{\max},\tag{1}$$

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corresponding to the *maximum* attainable visibility evaluated over the equivalency class of all spatial coherency matrices $\mathbf{G}_\mathrm{s}$ related through 2×2 spatial unitaries.

2.2. 4×4-Matrix description of the spatial-polarization Hilbert space

Proceeding to the case of a *vector* EM field, the correlations between the field components at points $\vec{r}_\mathrm{a}$ and $\vec{r}_\mathrm{b}$ are represented by a Hermitian, positive semi-definite 4×4 coherency matrix,

$$\mathbf{G}=\begin{pmatrix} G_\mathrm{aa}^\mathrm{HH} & G_\mathrm{aa}^\mathrm{HV} & G_\mathrm{ab}^\mathrm{HH} & G_\mathrm{ab}^\mathrm{HV} \\ G_\mathrm{aa}^\mathrm{VH} & G_\mathrm{aa}^\mathrm{VV} & G_\mathrm{ab}^\mathrm{VH} & G_\mathrm{ab}^\mathrm{VV} \\ G_\mathrm{ba}^\mathrm{HH} & G_\mathrm{ba}^\mathrm{HV} & G_\mathrm{bb}^\mathrm{HH} & G_\mathrm{bb}^\mathrm{HV} \\ G_\mathrm{ba}^\mathrm{VH} & G_\mathrm{ba}^\mathrm{VV} & G_\mathrm{bb}^\mathrm{VH} & G_\mathrm{bb}^\mathrm{VV} \end{pmatrix}=\begin{pmatrix} \mathbf{G}_\mathrm{aa} & \mathbf{G}_\mathrm{ab} \\ \mathbf{G}_\mathrm{ba} & \mathbf{G}_\mathrm{bb} \end{pmatrix}, \tag{2}$$

where $G_{kk'}^{jj'}=\langle E_j(\vec{r}_k)E_{j'}^*(\vec{r}_{k'})\rangle$, $G_{kk'}^{jj'}=(G_{k'k}^{j'j})^*$, the fields are normalized such that $\mathbf{G}$ has unity trace, $j,j'=\mathrm{H},\mathrm{V}$, and $k,k'=\mathrm{a},\mathrm{b}$ [20, 21]; $G_\mathrm{ab}^\mathrm{VH}$, for example, represents the two-point correlations between the V component at $\vec{r}_\mathrm{a}$ and the the H component at $\vec{r}_\mathrm{b}$. The matrix $\mathbf{G}$ can be viewed in block-diagonal form, where $\mathbf{G}_\mathrm{aa}$, $\mathbf{G}_\mathrm{ab}$, $\mathbf{G}_\mathrm{ba}$, and $\mathbf{G}_\mathrm{bb}$ are 2×2 polarization coherency matrices of the form $\mathbf{G}_{kk'}=\begin{pmatrix} G_{kk'}^\mathrm{HH} & G_{kk'}^\mathrm{HV} \\ G_{kk'}^\mathrm{VH} & G_{kk'}^\mathrm{VV}\end{pmatrix}$, where $k,k'=\mathrm{a},\mathrm{b}$. Here $\mathbf{G}_\mathrm{aa}$ and $\mathbf{G}_\mathrm{bb}$ are *Hermitian* polarization coherency matrices at $\vec{r}_\mathrm{a}$ and $\vec{r}_\mathrm{b}$, respectively, whereas $\mathbf{G}_\mathrm{ab}$ and $\mathbf{G}_\mathrm{ba}$ are the 2×2 cross-spectral density matrix for $\vec{r}_\mathrm{a}$ and $\vec{r}_\mathrm{b}$ [9] or the beam coherence-polarization (BCP) matrix [7] – and are *not* necessarily Hermitian; however $\mathbf{G}_\mathrm{ab}=\mathbf{G}_\mathrm{ba}^\dagger$.

Although these matrix blocks are *separately* well-known in coherence theory, their arrangement *together* in a 4×4 matrix is more convenient in many cases. In particular, it facilitates studying the field transformation under the influence of unitaries spanning the spatial and polarization DoFs, and it also enables a clear benchmarking of various proposed measures of spatial coherence and interference visibility for EM fields. Indeed, this 4×4 formulation is implicit in the tensor representation of partially coherent EM [22, 23], but it is nevertheless not regularly utilized.

A matrix that will be of utility is the diagonal form of $\mathbf{G}$. The real, positive eigenvalues of $\mathbf{G}$ are denoted $\{\lambda_j\}$, $j=1\ldots4$, $\sum_j\lambda_j=1$ and, without loss of generality, $\lambda_1\geq\lambda_2\geq\lambda_3\geq\lambda_4\geq0$,

$$\mathbf{G}^\mathrm{D}=\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}=\mathrm{diag}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}, \tag{3}$$

referred to hereon as the *canonical* diagonalized coherency matrix. The diagonalization can always be carried out via an appropriate 4×4 unitary $\hat{U}$, $\mathbf{G}^\mathrm{D}=\hat{U}\mathbf{G}\hat{U}^\dagger$. These eigenvalues can be interpreted as the weight of four orthogonal modes (polarized and spatially coherent fields) that are mixed to create the field represented by $\mathbf{G}$ [20]. For a coherent-polarized field $\{\lambda\}=\{1,0,0,0\}$ and for an incoherent-unpolarized field $\{\lambda\}=\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\}$ [24].

For a classical EM field, all the information about its second-order field correlations is encoded in $\mathbf{G}$ [25], which is measurable in its entirety via optical coherency matrix tomography – proposed theoretically in [21] and demonstrated experimentally in [26]. This coherency matrix is an element of the four-dimensional Hilbert space formed of a direct product of the two-dimensional Hilbert spaces associated with polarization and spatial DoFs described above. As such, $\mathbf{G}$ is isomorphic to the density matrix in quantum mechanics representing two-qubit states [27] – an analogy that has recently proven fruitful in optics [24, 28, 29].

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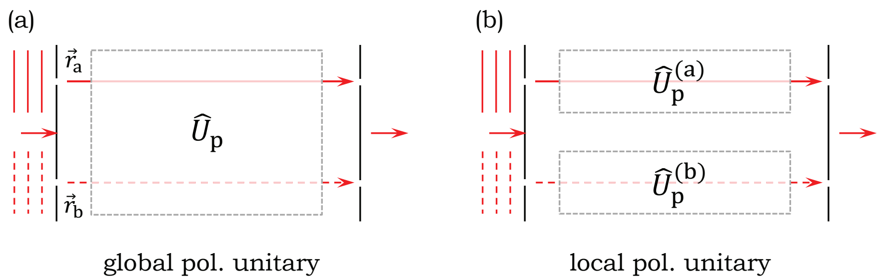

Fig. 1. (a) Global polarization unitary transformation $\hat{U}_p$ applied to both $\vec{r}_a$ and $\vec{r}_b$. (b) Local polarization unitaries $\hat{U}_p^{(a)}$ and $\hat{U}_p^{(b)}$ applied at $\vec{r}_a$ and $\vec{r}_b$, respectively.

2.3. Formulation of the problem studied here

In this paper, I consider the following problem: *what is the maximum double-slit visibility achievable when a partially coherent and partially polarized EM field undergoes the most general unitary transformation spanning both the polarization and spatial DoFs?* Answering this question requires first identifying the various families of unitaries operating on the Hilbert space of 4×4 coherency matrices and their impact on $\mathbf{G}$. Four classes of unitary transformations on the spatial and polarization DoFs of interest are listed:

1. A global polarization unitary encompassing $\vec{r}_a$ and $\vec{r}_b$; i.e., a spatially independent polarization unitary $\hat{U}_p$ (lossless birefringent device) covering both points [Fig. 1(a)]. The corresponding 4×4 transformation takes the form $\hat{U}=\hat{\mathbb{I}}_2\otimes\hat{U}_p=\begin{pmatrix}\hat{U}_p & \mathbf{0}\\ \mathbf{0} & \hat{U}_p\end{pmatrix}$, where $\hat{\mathbb{I}}_2$ is the 2×2 unity matrix and $\mathbf{0}$ is the 2×2 zero matrix, and $\mathbf{G}$ thus transforms according to
$$\mathbf{G}\rightarrow\hat{U}\mathbf{G}\hat{U}^{\dagger}=\begin{pmatrix}\hat{U}_p\mathbf{G}_{aa}\hat{U}_p^{\dagger} & \hat{U}_p\mathbf{G}_{ab}\hat{U}_p^{\dagger}\\ \hat{U}_p\mathbf{G}_{ba}\hat{U}_p^{\dagger} & \hat{U}_p\mathbf{G}_{bb}\hat{U}_p^{\dagger}\end{pmatrix}. \tag{4}$$
As such, $\hat{U}$ retains the block form of $\mathbf{G}$ and does not 'mix' the block matrices with each other. It will be shown that implementing such a unitary does *not* change the visibility V.

2. Different polarization unitaries $\hat{U}_p^{(a)}$ and $\hat{U}_p^{(b)}$ at $\vec{r}_a$ and $\vec{r}_b$, respectively, corresponding to the unitary $\hat{U}=\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes\hat{U}_p^{(a)}+\begin{pmatrix}0&0\\0&1\end{pmatrix}\otimes\hat{U}_p^{(b)}=\begin{pmatrix}\hat{U}_p^{(a)} & \mathbf{0}\\ \mathbf{0} & \hat{U}_p^{(b)}\end{pmatrix}$, which no longer separates into a direct product of spatial and polarization unitaries [Fig. 1(b)]. Thus, $\mathbf{G}$ transforms according to
$$\mathbf{G}\rightarrow\begin{pmatrix}\hat{U}_p^{(a)}\mathbf{G}_{aa}\hat{U}_p^{(a)\dagger} & \hat{U}_p^{(a)}\mathbf{G}_{ab}\hat{U}_p^{(b)\dagger}\\ \hat{U}_p^{(b)}\mathbf{G}_{ba}\hat{U}_p^{(a)\dagger} & \hat{U}_p^{(b)}\mathbf{G}_{bb}\hat{U}_p^{(b)\dagger}\end{pmatrix}. \tag{5}$$
It is critical to note that, once again, the block form of $\mathbf{G}$ is retained. Such a transformation *can* change V, and indeed this class of *local* polarization unitaries has been the focus of most studies investigating maximizing the double-slit visibility with EM fields to date [11,12]

3. Spatial unitaries that are independent of polarization, thus having the form $\hat{U}=\hat{U}_s\otimes\hat{\mathbb{I}}_2$,

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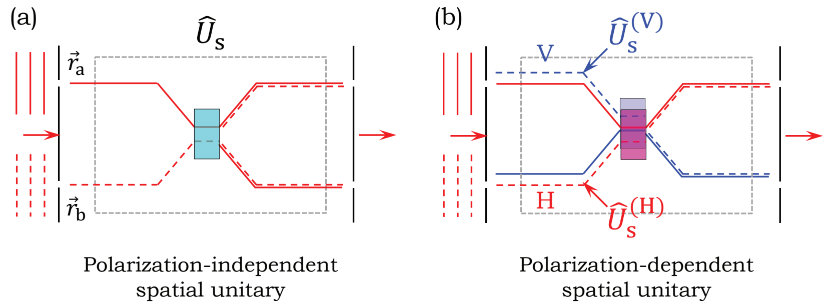

Fig. 2. (a) Spatial unitary transformation $\hat{U}_s$ that is polarization-*independent*, depicted as a generalized beam splitter [Eq. 6]. (b) Spatial polarization transformation that is polarization-*dependent* [Eq. 7], depicted as a polarizing beam splitter. The H and V polarization components undergo different spatial unitaries $\hat{U}_s^{(H)}$ and $\hat{U}_s^{(V)}$, respectively.

such as a symmetric beam splitter or coupler [Fig. 2(a)] with $\hat{U}_s = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$,

$$\hat{U}_{BS} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & i & 0\\ 0 & 1 & 0 & i\\ i & 0 & 1 & 0\\ 0 & i & 0 & 1\end{pmatrix}. \tag{6}$$

The utility of the 4×4 formulation of **G** becomes clear in this case. It is critical to note that unlike the polarization unitaries, this spatial unitary *mixes* the blocks in **G**. For example, starting from a diagonal coherency matrix $\mathbf{G} = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, whereupon $\mathbf{G}_{ab} = \mathbf{0}$, after applying $\hat{U}$ we have $\mathbf{G}_{ab} \neq \mathbf{0}$. This feature will be crucial in our analysis below. Most importantly, such transformations can change the value of V. These unitaries belong to the class of transformations considered by Zernike with respect to maximizing V for *scalar* fields [19].

4. Polarization-dependent spatial unitary transformations that introduce a spatial transformation that differs for each polarization component; e.g., $\hat{U} = \hat{U}_s^{(H)} \otimes \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \hat{U}_s^{(V)} \otimes \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. Here, $\hat{U}_s^{(H)}$ and $\hat{U}_s^{(V)}$ are the *spatial* transformations undergone by the H and V polarization components, respectively [Fig. 2(b)]. One example is a polarizing beam splitter in which the H polarization is transmitted and V is reflected, $\hat{U}_s^{(H)} = \mathbb{I}_2$ and $\hat{U}_s^{(V)} = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, leading to

$$\hat{U}_{PBS} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & i\\ 0 & 0 & 1 & 0\\ 0 & i & 0 & 0\end{pmatrix}, \tag{7}$$

Such unitaries mix the blocks of **G** in such as way as to convert coherence from one DoF to another [24] and thus can radically change the value of V. This class of transformations has *not* been considered in previous work on maximizing V.

More general unitaries can be formed as a cascade of elements from these four groups [Fig. 3]. The problem that is tackled in this paper is thus as follows: given a coherency matrix **G**, what is the maximum double-slit visibility attained after the transformation $\mathbf{G} \rightarrow \hat{U}\mathbf{G}\hat{U}^{\dagger}$, where $\hat{U}$ is the most general 4×4 unitary transformation?

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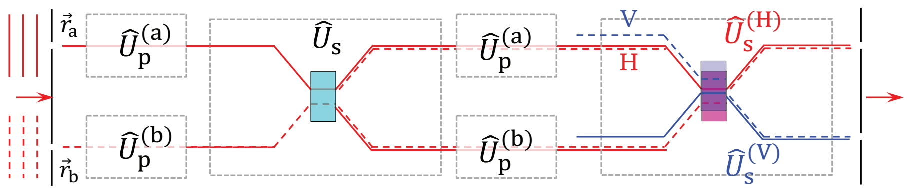

Fig. 3. General unitary transformation extending across the spatial and polarization DoFs formed as a cascade of the polarization and spatial unitaries in Fig. 1 and Fig. 2, respectively.

3. Review of previous work on the problem

Because double-slit interference is observed spatially, the visibility can be found by referring to the *reduced spatial coherency matrix* [21, 24] obtained by tracing over polarization in **G**,

$$\mathbf{G}_s^{(r)} = \begin{pmatrix} G_{aa}^{HH}+G_{aa}^{VV} & G_{ab}^{HH}+G_{ab}^{VV} \\ G_{ba}^{HH}+G_{ba}^{VV} & G_{bb}^{HH}+G_{bb}^{VV} \end{pmatrix}, \tag{8}$$

which describes the field spatial properties when the detectors are *in*sensitive to polarization. Note that diagonalizing $\mathbf{G}_s^{(r)}$ requires only 2×2 *spatial* unitaries, while diagonalizing **G** requires more general 4×4 *spatial-polarization* unitaries. The double-slit interference visibility is

$$V_0 = 2\left|G_{ab}^{HH}+G_{ab}^{VV}\right| = 2|\mathrm{Tr}\{\mathbf{G}_{ab}\}|, \tag{9}$$

which combines the results for the H and V components; that is, there is no influence from correlations between H and V, such as the element G_{ab}^{HV} of **G**. This result is related to the spectral degree of coherence as defined by E. Wolf in Ref. [9] and early on by Karczewski [30].

Applying a global polarization unitary $\hat{U}_p$ that introduces the map $\mathbf{G}_{ab} \to \hat{U}_p\mathbf{G}_{ab}\hat{U}_p^{\dagger}$ per Eq. 4 does *not* change V_0 because $\mathrm{Tr}\{\mathbf{G}_{ab}\} = \mathrm{Tr}\{\hat{U}_p\mathbf{G}_{ab}\hat{U}_p^{\dagger}\}$ [31]. On the other hand, applying *local* polarization unitaries $\hat{U}_p^{(a)}$ and $\hat{U}_p^{(b)}$ that introduce the mapping $\mathbf{G}_{ab} \to \hat{U}_p^{(a)}\mathbf{G}_{ab}\hat{U}_p^{(b)\dagger}$ per Eq. 5 *does* change V_0 because $\mathrm{Tr}\{\mathbf{G}_{ab}\} \neq \mathrm{Tr}\{\hat{U}_p^{(a)}\mathbf{G}_{ab}\hat{U}_p^{(b)\dagger}\}$. In other words, V_0 is not a unitary invariant, a fact that has prompted introducing an alternative definition for EM spatial coherence proposed in Refs. [32, 33] and called the 'electromagnetic degree of coherence' γ, where $\gamma^2 = \mathrm{Tr}\{\mathbf{G}_{ab}\mathbf{G}_{ab}^{\dagger}\}/(\mathrm{Tr}\{\mathbf{G}_{aa}\}\mathrm{Tr}\{\mathbf{G}_{bb}\})$. This quantity represents the correlation between all the pairs of components of the fields at $\vec{r}_a$ and $\vec{r}_b$ and is invariant under local polarization unitaries – however, γ is not directly related to the visibility, and other measurements are required to determine it [32, 33]. Maximizing $V = 2|\mathrm{Tr}\{\hat{U}_p^{(a)}\mathbf{G}_{ab}\hat{U}_p^{(b)\dagger}\}|$ over the span of all *local polarization unitaries* is equivalent to finding the so-called Ky-Fan 1-norm [34] of $\mathbf{G}_{ab}$, which yields

$$V_{LPU} = 2(\mu_1+\mu_2) = 2\sqrt{\mathrm{Tr}\{\mathbf{G}_{ab}\mathbf{G}_{ab}^{\dagger}\} + 2|\det\{\mathbf{G}_{ab}\}|}, \tag{10}$$

where μ_1 and μ_2 are the singular values of $\mathbf{G}_{ab}$ [11, 12], while a unity-trace for **G** is maintained. Other measures have been introduced that rely on non-unitary local polarization transformations and thus lead to loss of energy; these will be described in the Discussion Section.

Common to all previous efforts on maximizing the double-slit interference visibility or identifying measures for spatial coherence is reliance on $\mathbf{G}_{ab}$ (e.g., Eq. 9 and Eq. 10; see the definition of the 'complex degree of mutual coherence' [35] that also requires a non-zero $\mathbf{G}_{ab}$). A class of EM fields that evades these analyses is that having the block-diagonal representation of the coherency matrix $\mathbf{G} = \begin{pmatrix} \mathbf{G}_{aa} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{bb} \end{pmatrix}$. The zero off-diagonal blocks indicate that the fields at $\vec{r}_a$

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and $\vec{r}_\mathrm{b}$ are statistically independent; i.e., spatially *in*coherent fields that are partially polarized. At one extreme, $\mathbf{G}_\mathrm{aa}=\mathbf{G}_\mathrm{bb}\propto\begin{pmatrix}1&0\\0&0\end{pmatrix}$, in which case $\mathbf{G}$ corresponds to a *scalar* field that is spatially *in*coherent. At the other extreme, $\mathbf{G}_\mathrm{aa}=\mathbf{G}_\mathrm{bb}\propto\begin{pmatrix}1&0\\0&1\end{pmatrix}$, corresponding to *un*polarized spatially incoherent light. Across this continuum of states of coherence maintaining $\mathbf{G}_\mathrm{ab}=\mathbf{0}$, all the measures described above necessarily yield $V=0$ – implementing polarization unitaries at $\vec{r}_\mathrm{a}$ and $\vec{r}_\mathrm{b}$ notwithstanding. Such an outcome may be expected since the field is spatially *in*coherent. Nevertheless, such fields may still display high-visibility double-slit fringes – even reaching $V=1$ – once unitaries that span *both* the spatial and the polarization DoFs are employed.

4. Derivation of V_max

I now turn to the titular question and determine the maximal visibility $V_\mathrm{max}=\max\{V_0\}$ attainable by an EM field under arbitrary 4×4 spatial-polarization unitaries $\hat{U}$, $\hat{U}\hat{U}^\dagger=\mathbb{I}_4$ with elements $\{u_{ij}\}$, $i,j=1\ldots4$. Starting from $\mathbf{G}^\mathrm{D}=\mathrm{diag}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$, the most general coherency matrix is $\mathbf{G}=\hat{U}\mathbf{G}^\mathrm{D}\hat{U}^\dagger$ [31]. Let us define the quantity $X=G_\mathrm{aa}^\mathrm{HH}+G_\mathrm{aa}^\mathrm{VV}-G_\mathrm{bb}^\mathrm{HH}-G_\mathrm{bb}^\mathrm{VV}$. It can be shown that $X=X_1+X_2-X_3-X_4$, where $X_k=\sum_{j=1}^{4}\lambda_j|u_{jk}|^2$. Referring to Eq. 8 and the definition of D_s for a scalar field in Eq. 1, it is clear that $V_\mathrm{max}=\max\{X\}$ evaluated over all possible $\hat{U}$. Since all the values entering into X are positive real numbers, maximizing X requires maximizing X_1+X_2 and minimizing X_3+X_4 subject to the unitarity of $\hat{U}$. Because the eigenvalues $\{\lambda_j\}$ are non-negative and arranged in descending value, then X_1+X_2 reaches a maximum of $\lambda_1+\lambda_2$ when $u_{31}=u_{41}=u_{32}=u_{42}=0$. Likewise, X_3+X_4 simultaneously reaches a minimum of $\lambda_3+\lambda_4$ with $u_{13}=u_{14}=u_{23}=u_{24}=0$. Thus $\mathbf{G}$ is block-diagonal $\begin{pmatrix}\mathbf{G}_\mathrm{aa}&\mathbf{0}\\\mathbf{0}&\mathbf{G}_\mathrm{bb}\end{pmatrix}$, with $\mathbf{G}_\mathrm{aa}$ and $\mathbf{G}_\mathrm{bb}$ related to $\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}$ and $\begin{pmatrix}\lambda_3&0\\0&\lambda_4\end{pmatrix}$, respectively, via 2×2 local polarization unitaries.

The question posed at the outset can now be answered. Starting from a coherency matrix $\mathbf{G}$, the maximum double-slit interference visibility attainable by the EM field is given by:

$$V_\mathrm{max}=\lambda_1+\lambda_2-\lambda_3-\lambda_4=1-2(\lambda_3+\lambda_4). \tag{11}$$

This equation is the central result of the Letter.

An unexpected result can be stated immediately. EM fields characterized by coherency matrices possessing three or four non-zero eigenvalues, $\{\lambda\}=\{\lambda_1,\lambda_2,\lambda_3,0\}$ and $\{\lambda\}=\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$, respectively, have $V_\mathrm{max}<1$. On the other hand, EM fields whose coherency matrices possess one or two non-zero eigenvalues, $\{\lambda\}=\{1,0,0,0\}$ and $\{\lambda\}=\{\lambda_1,\lambda_2,0,0\}$, respectively (i.e., the two smallest eigenvalues λ_3 and λ_4 vanish) – *always* attain $V_\mathrm{max}=1$. The first class of EM fields $\{\lambda\}=\{1,0,0,0\}$ corresponds to *coherent fully polarized* fields, whereas the second class of EM fields $\{\lambda\}=\{\lambda_1,\lambda_2,0,0\}$ corresponds to *partially coherent partially polarized* fields that nevertheless can exhibit full-visibility double-slit interference fringes $V_\mathrm{max}=1$. This latter class is of particular interest since it encompasses scalar fields that lack all coherence, and yet full interference visibility is predicted.

5. Discussion

5.1. Examples

I consider here a few examples of EM fields to clarify the concepts discussed thus far:

$$\mathbf{G}_1=\frac{1}{10}\begin{pmatrix}2&0&1&0\\0&2&0&1\\1&0&3&0\\0&1&0&3\end{pmatrix},\quad\mathbf{G}_2=\frac{1}{2}\begin{pmatrix}1&0&0&a\\0&0&0&0\\0&0&0&0\\a^*&0&0&1\end{pmatrix},\quad\mathbf{G}_3=\frac{1}{4}\begin{pmatrix}1&a&0&0\\a^*&1&0&0\\0&0&1&b\\0&0&b^*&1\end{pmatrix},\tag{12}$$

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where $0 \leq |a|, |b| \leq 1$ and $|a| \geq |b|$ without loss of generality. These three matrices are Hermitian and positive semi-definite, and thus represent genuine coherency matrices for all values of a and b.

The first example $\mathbf{G}_1$ corresponds to a field that has unequal field amplitudes at $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$, is partially coherent *spatially*, but is *unpolarized* at both $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$. For such a field, $V_0 = 2/5$. Furthermore, $V_{\mathrm{LPU}} = V_0$ because $\mathbf{G}_{\mathrm{ab}} \propto \mathbb{I}_2$. The eigenvalues of $\mathbf{G}_1$ are $\{\lambda\} = \frac{1}{4}\{1+1/\sqrt{5}, 1+1/\sqrt{5}, 1-1/\sqrt{5}, 1-1/\sqrt{5}\}$, leading to $V_{\max} = \sqrt{5}/5 > V_0$.

The second example $\mathbf{G}_2$ corresponds to a partially coherent field that has orthogonal polarizations at $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$. The value of a determines the spatial correlations between these two orthogonal polarization components. The EM field represented by $\mathbf{G}_2$ yields $V_0 = 0$ because $\mathrm{Tr}\{\mathbf{G}_{\mathrm{ab}}\} = 0$. Because $\mathrm{Tr}\{\mathbf{G}_{\mathrm{ab}}\mathbf{G}_{\mathrm{ab}}^{\dagger}\} = |a|^2/4$, local polarization unitaries can nevertheless increase the visibility to $V_{\mathrm{LPU}} = |a| \leq 1$ [Eq. 10], as can be expected since $|a|$ determines the spatial coherence once the field polarizations at $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$ are made parallel to each other (e.g., via a wave plate at $\vec{r}_{\mathrm{a}}$). However, the eigenvalues of $\mathbf{G}_2$ are $\{\lambda\} = \frac{1}{2}\{1+|a|, 1-|a|, 0, 0\}$, which yield $V_{\max} = 1$ *independently* of the value of a. In other words, there exists a 4×4 polarization-spatial unitary transformation that transforms the coherence matrix into a form that will yield unity-visibility double-slit interference fringes – even when $a = 0$ and the fields at $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$ are completely uncorrelated.

The third example $\mathbf{G}_3$ represents light that is partially polarized at $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$ with different degrees of polarization, but is spatially incoherent (the fields at $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$ are statistically independent, $\mathbf{G}_{\mathrm{ab}} = \mathbf{0}$). All the measures of spatial coherence or double-slit visibility discussed earlier predict zero-visibility for such a field. The eigenvalues of $\mathbf{G}$ are $\{\lambda\} = \frac{1}{4}\{1+|a|, 1+|b|, 1-|b|, 1-|a|\}$, and thus $V_{\max} = (|a|+|b|)/2 = (D_{\mathrm{p}}^{(\mathrm{a})} + D_{\mathrm{p}}^{(\mathrm{b})})/2$; that is, the maximum visibility is determined by the degrees of polarization $D_{\mathrm{p}}^{(\mathrm{a})} = |a|$ and $D_{\mathrm{p}}^{(\mathrm{b})} = |b|$ at $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$, respectively, even though the field is spatially incoherent. Indeed, $V_{\max}$ is guaranteed to be non-zero *as long as the field is at least partially polarized at one point*, with $V_{\max} = 1$ when the field is fully polarized at both points (the field need not be scalar and the polarization at $\vec{r}_{\mathrm{a}}$ can be different from that at $\vec{r}_{\mathrm{b}}$). I describe in Section 6 a specific example of how to convert the field described by $\mathbf{G}_3$ to a form that exhibits this finite visibility.

5.2. Comparison to results relying on non-unitary transformations

The visibility may of course be increased via *non*-unitary transformations that involve filtering or projecting either or both DoFs, which reduce the energy. The use of such transformations involves an element of arbitrariness, in contrast to reliance on unitary transformations that conserve energy. Nevertheless, some interesting studies have been reported along this vein, and I compare them here to the measure $V_{\max}$ introduced in this paper.

(1) The work by Réfrégier and Goudail on so-called ‘intrinsic degrees of coherence’ [36] does not give a closed-form expression for the identified unitary invariants $0 \leq \mu_{\mathrm{S}}, \mu_{\mathrm{I}} \leq 1$ ($\mu_{\mathrm{S}} \geq \mu_{\mathrm{I}}$); instead, an *algorithm* for extracting them from $\mathbf{G}$ is put forth [14]: (1) local polarization unitaries diagonalize $\mathbf{G}_{\mathrm{aa}}$ and $\mathbf{G}_{\mathrm{bb}}$; (2) the eigenvalues of $\mathbf{G}_{\mathrm{aa}}$ and $\mathbf{G}_{\mathrm{bb}}$ are ‘equalized’ by implementing local non-singular Jones matrices, which are *not* unitary; and (3) implementing new local polarization unitaries to diagonalize $\mathbf{G}_{\mathrm{ab}}$. The resulting coherency matrix has the form

$$\mathbf{G} = \frac{1}{4}\begin{pmatrix} 1 & 0 & \mu_{\mathrm{S}} & 0 \\ 0 & 1 & 0 & \mu_{\mathrm{I}} \\ \mu_{\mathrm{S}} & 0 & 1 & 0 \\ 0 & \mu_{\mathrm{I}} & 0 & 1 \end{pmatrix}. \tag{13}$$

An implicit assumption in this approach is that the power at $\vec{r}_{\mathrm{a}}$ is equal to that at $\vec{r}_{\mathrm{b}}$. Reaching a condition of equal power at $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$ requires either further filtering or a *spatial* transformation.

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Critically, it appears that reaching the form in Eq. 13 starting from an arbitrary field *necessitates* non-unitary transformations. In the form of $\mathbf{G}$ given in Eq. 13, $V_0 = V_{\mathrm{LPU}} = (\mu_{\mathrm{S}} + \mu_{\mathrm{I}})/2$, and the eigenvalues are $\{\lambda\} = \frac{1}{4}\{1+\mu_{\mathrm{S}}, 1+\mu_{\mathrm{I}}, 1-\mu_{\mathrm{I}}, 1-\mu_{\mathrm{S}}\}$ so that $V_{\max} = V_0$.

(2) The analysis by Luis [15] suggests the definition $V_{\mathrm{L}} = \frac{\lambda_1 - \lambda_4}{\lambda_1 + \lambda_4}$. This expression is obtained by first transforming the field via a spatial-polarization unitary to the canonical diagonal form, followed by *projecting* or *filtering* out the modes associated with the eigenvalues λ_2 and λ_3 thus eliminating a fraction $\lambda_2 + \lambda_3$ of the total power (normalized to unity). On the other hand, after such a projection, the definition in Eq. 11 gives $V_{\max} = 1 \geq V_{\mathrm{L}}$. The analysis presented here suggests an *optimal* filtering methodology to maximize V: filter out the modes associated with λ_3 and λ_4 (instead of λ_2 and λ_3). This procedure has two advantages: a smaller fraction of energy is lost since $\lambda_3 + \lambda_4 \leq \lambda_2 + \lambda_3$ and the resulting visibility is *always* $V_{\max} = 1$.

(3) Another approach to determining the double-slit visibility involves a generalized form of the Fresnel-Arago interference laws [13], but this requires *first* placing linear polarizers at $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$. Within our approach, placing linear polarizers at $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$ always produces $V_{\max} = 1$ independently of the state of coherence (e.g., the examples of $\mathbf{G}_2$ and $\mathbf{G}_3$ in Eq. 12).

6. Coherency conversion

I illustrate here with an example the conversion of coherence between the polarization and spatial DoFs. Consider a spatially *in*coherent *scalar* field with equal amplitudes at $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$, $\mathbf{G}_1 = \frac{1}{2}\mathrm{diag}\{1,0,1,0\}$ – for which $\mathbf{G}_{\mathrm{ab}} = \mathbf{0}$ and thus $V_0 = V_{\mathrm{LPU}} = 0$ as expected for a field lacking any spatial coherence [Fig. 4(a)]. Nevertheless, Eq. 11 gives $V_{\max} = 1$. One sequence of *non-commuting* spatial-polarization unitaries that transforms $\mathbf{G}_1$ (spatially incoherent, polarized) to $\mathbf{G}_4$ (spatially coherent, unpolarized) that displays full visibility is given by

$$\underbrace{\mathbf{G}_1 = \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}}_{\mathbf{G}_{\mathrm{s}}^{(\mathrm{r})} = \frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} \to V_0 = 0} \xrightarrow{\hat{U}_{12}} \underbrace{\mathbf{G}_2 = \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}}_{\mathbf{G}_{\mathrm{s}}^{(\mathrm{r})} = \frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} \to V_0 = 0} \xrightarrow{\hat{U}_{23}} \underbrace{\mathbf{G}_3 = \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}}_{\mathbf{G}_{\mathrm{s}}^{(\mathrm{r})} = \begin{pmatrix}1&0\\0&0\end{pmatrix} \to V_0 = 1} \xrightarrow{\hat{U}_{34}} \underbrace{\mathbf{G}_4 = \frac{1}{4}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix}}_{\mathbf{G}_{\mathrm{s}}^{(\mathrm{r})} = \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix} \to V_0 = 1}, \tag{14}$$

where $\hat{U}_{12}$ corresponds to a half-wave plate placed at $\vec{r}_{\mathrm{b}}$ that rotates the polarization from H to V, $\hat{U}_{23}$ is a polarizing beam splitter [Eq. 7], and $\hat{U}_{34}$ is a beam splitter [Eq. 6] – with the latter two having adjusted phases. Note that $\mathbf{G}_2$ describes a partially coherent field where polarization is now correlated with position, such that the initially separable spatial and polarization DoFs $\mathbf{G}_1 = \frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}_{\mathrm{s}} \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix}_{\mathrm{p}}$ become intertwined after $\hat{U}_{12}$, and $\mathbf{G}_2$ is no longer factorizable (the subscripts 's' and 'p' refer to the spatial and polarization DoF's, respectively). The polarizing beam splitter $\hat{U}_{23}$ combines the fields from $\vec{r}_{\mathrm{a}}$ and $\vec{r}_{\mathrm{b}}$ to produce an unpolarized field fully localized at $\vec{r}_{\mathrm{a}}$. Here $\mathbf{G}_3 = \begin{pmatrix}1&0\\0&0\end{pmatrix}_{\mathrm{s}} \otimes \frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}_{\mathrm{p}}$ is again separable in its DoFs, the field is now spatially coherent but unpolarized. This separability is critical for the concept of field-protection via coherency conversion discussed below. The beam splitter $\hat{U}_{34}$ splits the field at $\hat{r}_{\mathrm{a}}$ into equal-amplitude spatially coherent fields at $\hat{r}_{\mathrm{a}}$ and $\hat{r}_{\mathrm{b}}$, $\mathbf{G}_4$. Therefore, a polarized but spatially incoherent field $\mathbf{G}_1$ that displays zero interference visibility has thus been transformed to a spatially coherent but unpolarized field $\mathbf{G}_4$ that displays full visibility. I call this process 'coherency conversion' [Fig. 4(b)]. The procedure is fully reversible and there has been no optical energy lost.

The same approach applies not only to two points in a scalar incoherent field, but to the entire field via a similar sequence of unitaries, as shown in Fig. 4(c). The beam coherency matrix (BCP) [7] of the initial field is $\mathbf{G}_1 = \begin{pmatrix}f(x_1,x_2)&0\\0&0\end{pmatrix}$, where $f(x_1,x_2) = I(x_1)\delta(x_1 - x_2)$

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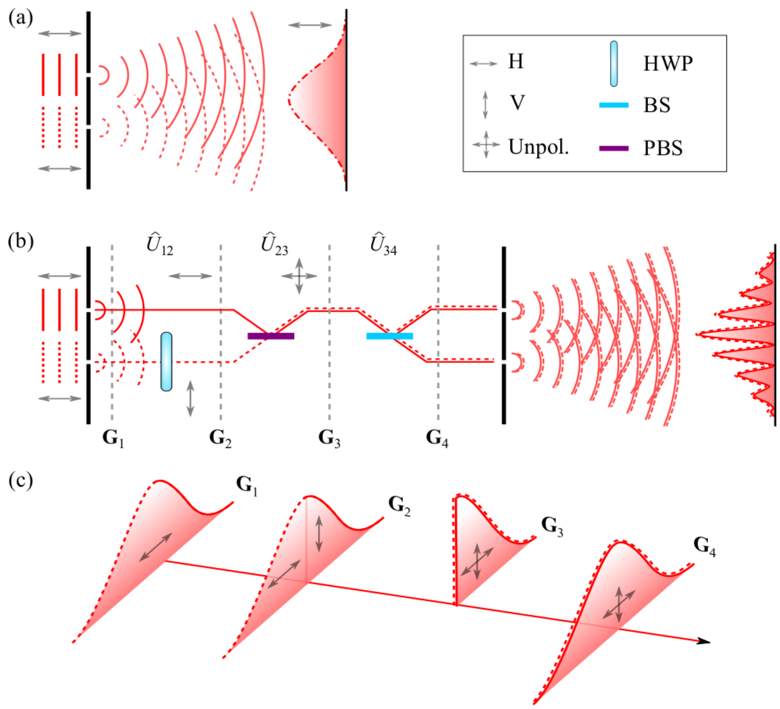

Fig. 4. (a) Two-points in a spatially *in*coherent *scalar* field $\mathbf{G}_1$ produce no interference. (b) Reversibly transforming the field in (a) to produce full-visibility interference fringes via a succession of unitaries: $\hat{U}_{12}$ rotates the polarization at $\vec{r}_\mathrm{b}$ by $\frac{\pi}{2}$, $\mathbf{G}_2$; $\hat{U}_{23}$ is a polarizing beam splitter that combines the field at $\vec{r}_\mathrm{a}$ and $\vec{r}_\mathrm{b}$ to produce unpolarized light at $\vec{r}_\mathrm{a}$, $\mathbf{G}_3$; and, finally, a non-polarizing beam splitter $\hat{U}_{34}$ produces a spatially *coherent* – albeit *unpolarized* – field $\mathbf{G}_4$. (c) Reversibly transforming a scalar incoherent field to produce full visibility interference fringes using a sequence of unitaries similar to (b). HWP: half-wave plate; BS: beam splitter; PBS: polarization BS.

is a scalar coherency function and $I(x)$ is the intensity distribution assumed for simplicity to be even $I(x_1) = I(-x_1)$. Polarization in one half of the wavefront is rotated from H to V, $\mathbf{G}_2 = \begin{pmatrix} f_+(x_1,x_2) & 0 \\ 0 & f_-(x_1,x_2) \end{pmatrix}$, where $f_\pm(x_1,x_2) = f(x_1,x_2)h(\pm x_1)$ and $h(x)$ is the Heaviside unit step function: $h(x) = 1$ for $x \geq 0$ and is zero otherwise. The second beam-half is combined with the first via a polarizing beam splitter to produce an unpolarized asymmetric beam, $\mathbf{G}_3 = 2f(x_1,x_2)h(x_1)\cdot\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The beam is then split into two halves again, resulting now in a symmetrized unpolarized beam, $\mathbf{G}_4 = f_\mathrm{s}(x_1,x_2)\cdot\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $f_\mathrm{s}(x_1,x_2) = I(x_1)\{\delta(x_1-x_2)+\delta(x_1+x_2)\}$, in which every pair of points $x_1 = -x_2$ symmetrically positioned around the central axis are now mutually coherent and thus produce Young's interference fringes with full visibility.

The above-described methodology suggests an approach for protecting a DoF of the EM field during propagation in a medium that introduces random fluctuations to this DoF. For example, consider transmitting a particular state of polarization through a depolarizing medium that is nevertheless spatially uniform. Making use of a spatially incoherent beam, coherence is first reversibly migrated from the polarization to the spatial DoF, rendering the beam unpolarized but spatially coherent while encoding the polarization state in the spatial DoF. The beam remains unpolarized after traversing the depolarizing medium and the initial polarization is finally retrieved by reversing the coherency-conversion process.

Is is instructive to view the procedure outlined above in light of the recently developed concept of 'classical entanglement' [24]. Initially, the field has independent spatial and polarization DoFs, as clear from the separability of the coherence matrix $\mathbf{G}_1$. The entropy of the spatial DoF is maximal (spatially uncorrelated or incoherent) whereas that of polarization is minimal (pure polarization).The impact of the HWP is to correlate the two DoFs: each point is now associated with a different polarization state, as seen in $\mathbf{G}_2$. At this point the entropy is distributed between the two DoFs. The PBS returns the field to a state where the two DoFs are independent and the coherency matrix $\mathbf{G}_3$ is once again separable. However, the entropy of the spatial DoF is now minimal and that of polarization is maximal. In previous studies of classical entanglement, the field examined was usually coherent and the impact of correlations between the DoFs was investigated. In contrast, the fields examined here are partially coherent, which indicates that the utility of the quantum-information-theoretic formulation exploited in studies of classical entanglement is readily extended to partially coherent classical fields.

7. Conclusion

In conclusion, I have developed a definitive answer to the question: what is the maximum visibility of Young's double-slit interference that may be attained by an EM field subject only to the most general reversible, unitary, energy-conserving transformations? By treating the spatial and polarization DoFs of the EM field symmetrically, a simple expression for the maximum interference visibility is obtained subject to arbitrary *spatial-polarization* unitary transformations. This visibility is an intrinsic invariant of the EM field and is evaluated in terms of the eigenvalues of the 4×4 spatial-polarization coherency matrix. The analysis presented reveals that the class of *scalar spatially incoherent fields can always exhibit unity interference visibility* from any two points upon implementing the appropriate spatial-polarization transformation that engenders coherency conversion between these two DoFs. That is, there exist unitary transformations that reversibly convert – with no loss in energy – a scalar field lacking any spatial coherence and thus exhibits no interference fringes to a spatially coherent but unpolarized field that exhibits full interference visibility.

Only two transverse polarization components of the EM field have been taken into account here. When considering the more general case of three polarization components, the analyses in Refs. [37–40] must be taken into consideration. Finally, the results presented here all pertain to the visibility of fringes observed in a Young's double-slit interference experiment. However, the methodology employed is based on treating the spatial and polarization DoFs symmetrically on the same footing. Therefore, it should be clear that these results similarly apply to polarimetry based on the polarization DoF.

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