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Even-order dispersion solitons: A pedagogical note

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ABSTRACT

In conventional temporal optical solitons the effect of the nonlinearity is balanced by that of quadratic dispersion. In recent work we have considered solitons that balance the nonlinearity with pure higher, even orders of dispersion or with combinations of different even dispersion orders. Here we highlight the key differences and similarities between these novel solitons and conventional solitons and give arguments for the existence of this infinite set of solitons.

1. Introduction

At an intuitive level, temporal solitons arise from the balance of nonlinear effects and the effects of dispersion, leading to a pulse that propagates without changing shape. In their simplest manifestation, the positive nonlinear effect is the Kerr effect, whereby the refractive index increases linearly with intensity, whereas the dispersion is *anomalous* and quadratic, so that the inverse group velocity is a linearly decreasing function of frequency, at least over the relevant range of frequencies (i.e., over the bandwidth of the pulse). This combination leads to "Nonlinear Schrödinger" (NLS) solitons. That the combination of a positive Kerr effect and negative quadratic dispersion can lead to the formation of solitons when light pulses propagate through an optical fiber, was first pointed out theoretically 50 years ago in a seminal paper by Hasegawa and Tappert [1]. This was followed in 1980 by its experimental confirmation by Mollenauer, Stolen and Gordon [2].

Hasegawa and Tappert showed that electric field envelope ψ of a short light pulse propagating in an optical fiber satisfies the nonlinear Schrödinger equation [1,3]

$$i\frac{\partial\psi}{\partial z} - \frac{\beta_2}{2!}\frac{d^2\psi}{dT^2} + \gamma|\psi|^2\psi = 0, \quad (1)$$

where z is the spatial propagation parameter and $T = t - z/v_g$ is time in a frame that moves at the group velocity v_g of the pulse. Further, β_2 is the quadratic dispersion and γ is the nonlinear parameter of the waveguide. Provided that β_2 and γ have opposite signs, the nonlinear Schrödinger equation has soliton solutions in the form of a hyperbolic secant [1,3]

There is a large literature on solitons with non-Kerr nonlinear effects, in which the relationship between the intensity and refractive index is more complicated than being merely linear [4–6]. Such models for the refractive index can, for example, represent the onset of saturation of the nonlinear contribution to the refractive index or damage to the material [7–9] or the finite time response of the nonlinear medium [10]. Historically, there have been far fewer investigations in which the limitation to quadratic dispersion has been relaxed. These include work in which higher orders of dispersion are included [11–16], but in most of these studies the quadratic dispersion nonetheless dominates. In recent years we and our co-workers and others have investigated theoretically and experimentally solitons with a Kerr nonlinearity, but with dispersion in which the quadratic term does not dominate or is even absent. This has lead to the discovery of an infinite hierarchy of solitons arising from pure-even-orders of dispersion and nonlinearity [17–20], as well as hybrid solutions [21–23], multipeak solutions [24,25], dark solitons [26,27], as well as solitons in cavities [27–30] and in fiber lasers [31–33].

Naturally, finding waveguides with dominant high-even orders of dispersion requires thought and careful design. Nonetheless, several realistic platforms are already available to experimentally study these novel kind of solitons. Pure-quartic solitons were originally observed in a planar photonic crystal waveguide [17]. Subsequently, pure high, even-order dispersion solitons were experimentally demonstrated in a fiber laser cavity with a reconfigurable dispersion lumped element [34, 35], up to dispersion of 10th order. Realistic designs with dominant high, even-order dispersion have been proposed in photonic crystal fibers [36], and microresonators [37,38].

The aim of this paper is to point out differences and similarities between the soliton solutions of the nonlinear Schrödinger equation and those of systems with more general dispersion relations, but still the simple Kerr nonlinearity, and to provide educational physical explanations to build a proper understanding of these recently discovered solitons. We consider five different aspects: (i) the energy and power scaling; (ii), the shape of the soliton; (iii) the presence/absence of

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Galilean invariance of the solitons; (iv) a phase compensation argument that physically illustrates the formation of solitons; and (v), the effective lengths associated with high orders of dispersion.

2. Background

In this paper we consider a conventional, third order Kerr nonlinearity, according to which

$$n(I) = n_l + n^{(2)} I, \tag{2}$$

where $n(I)$ is the intensity dependent refractive index, n_l is the low-intensity refractive index and $n^{(2)}$ is the nonlinear refractive index, which we will take to be positive. This means that, in the absence of other effects, by Eq. (2) the intensity gradient of a pulse causes a refractive index gradient, in turn leading to a phase gradient $\varphi(t)$ upon propagation. A phase gradient entails a frequency gradient through $\delta\omega = -\partial\varphi/\partial t$. For a positive nonlinearity this causes the leading edge to shift to the red whereas the trailing edge shifts to the blue [3,39].

In the most general case, the dispersion relation, the relationship between the frequency ω and the wavenumber β in a neighborhood around frequency ω_0 in the absence of nonlinear effects can be written as

$$\beta = \beta_0 + \beta_1(\omega-\omega_0) + \frac{\beta_2}{2!}(\omega-\omega_0)^2 + \frac{\beta_3}{3!}(\omega-\omega_0)^3 + \frac{\beta_4}{4!}(\omega-\omega_0)^4 + \cdots \tag{3}$$

where the $\beta_m = \partial^m\beta/\partial\omega^m$. The fixed wavenumber β_0 can be neglected as it leads to a fixed phase $\beta_0 L$ upon propagation over a distance L. In the second term on the right-hand side the factor β_1 represents the inverse group velocity $\partial\beta/\partial\omega = v_g^{-1}$ at ω_0. The associated term represents a pulse propagating at this group velocity without changing shape. This term can be dropped if we choose a frame moving at this speed. The lowest nontrivial term in Eq. (3) is thus the β_2 term, which represents quadratic dispersion. When higher order terms are neglected then the dispersion relation is a parabola and the inverse group velocity is a linear function of frequency. Since we take $\beta_2 < 0$ the inverse group velocity decreases linearly with frequency.

We now consider more general dispersion relations in which the inverse group velocity decreases monotonically with frequency, but not necessarily linearly. We further, for convenience, take the dispersion relation to be symmetric, so that dispersion relation Eq. (3) only has symmetric terms. In that case the nonlinear Schrödinger equation Eq. (1) generalizes to the generalized nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial z} + \sum_{m=2,4,6\ldots}^{M} (-1)^{m/2}\frac{\beta_m}{(m)!}\frac{\partial^m\psi}{\partial T^m} + \gamma|\psi|^2\psi = 0. \tag{4}$$

where M is the highest order of dispersion that is present.

3. Soliton formation

As discussed in Section 2, in the absence of other effects, the Kerr nonlinearity causes the leading edge of a pulse to shift to the red and the trailing edge to the blue. The formation of a stable pulse requires these newly generated frequencies to propagate toward the center of the pulse; in other words, the red frequencies need to slow down and the blue frequencies need to speed up. This means that inverse group velocity must decrease monotonically with frequency (so that the group velocity itself increases with frequency). From Eq. (3) the inverse group velocity $v_g^{-1} = \partial\beta/\partial\omega$ can be written as

$$\frac{1}{v_g} \equiv \frac{\partial\beta}{\partial\omega} = \beta_1 + \beta_2(\omega-\omega_0) + \frac{\beta_3}{2}(\omega-\omega_0)^2 + \frac{\beta_4}{6}(\omega-\omega_0)^3 + \cdots. \tag{5}$$

In conventional nonlinear Schrödinger solitons $\beta_2 < 0$ and higher order dispersion coefficients are negligible [1,3,39]; in this case the electric field envelope satisfies the nonlinear Schrödinger equation Eq. (1). However, there are of course many dispersion relations leading to a monotonically decreasing inverse group velocity with frequency. In these more general cases the electric field envelope satisfies Eq. (4). The purpose of this paper is to compare the soliton solutions to these two equations.

To bring some structure to the discussion we distinguish two cases. In the case of *pure dispersion* we take all but one of the dispersion coefficients to be nonzero, i.e., $\beta_m = 0$ for $m < M$ and $m > M$, and $\beta_M < 0$. For $M = 2$ this reverts to the nonlinear Schrödinger solitons, whereas $M = 4$ results in *Pure-Quartic Solitons* [17]. For $M > 2$, we generically refer to the resulting solitons as pure-high, even-order dispersion solitons (PHEODS) [19]. In contrast, in the presence of *mixed dispersion* all β_m for even $m \le M$ can be non-zero, and are such that v_g^{-1} monotonically decreases.

Since solitons do not change their shape upon propagation, they are solutions of Eq. (4) of the form $\psi(T,z) = e^{i\mu z}u(T)$ where μ is the nonlinear contribution to the propagation constant. The function $u(T)$ then satisfies the ordinary differential equation

$$-\mu u + \sum_{m=2,4,6\ldots}^{M} (-1)^{m/2}\frac{\beta_m}{(m)!}\frac{d^m u}{dT^m} + \gamma u^3 = 0. \tag{6}$$

We have further used that u is real—we discuss this further below. Since μ corresponds to the nonlinear contribution to the propagation constant, it is related to the peak power of the soliton. In general, therefore, $\mu \propto \gamma P$, where the proportionality constant is $1/2$ for conventional nonlinear Schrödinger solitons and typically is of order unity [19].

It is analytically known that the soliton solutions $u(T)$ for pure, quadratic dispersion are real [3]. Based on numerical results, the same is true for even-order soliton solutions, up to order $M = 16$ [35]. This means that for such solitons the phase is constant (i.e., they are *unchirped*). The reason is that in the presence of these even orders of dispersion, the flow of energy can be written as products of $d(\arg\{u\})/dT$, and its higher derivatives, with $|u|$ and its derivatives [40]. Now in order for the solution to be stationary it is necessary for the intra-pulse energy flow to vanish. This can be guaranteed if $\arg\{u\}$ is constant, i.e., if $u(T)$ has a constant phase and can thus be made real. We surmise the same to be true for any higher, even-order dispersion. We also have analytic and numerical results that this argument holds for solitons with mixed, even order of dispersion [22,41].

In the presence of β_3 this result no longer holds; the energy flow contains contributions of the form $2u(d^2u/dT^2) - (du/dT)^2$, which remains is nonzero, even in the absence of chirp. As shown in the Appendix, the same is true in the presence of any, odd-order dispersion $M > 3$. Thus solitons in the presence of any odd order of dispersion may exist, but they must be chirped. In the presence of cubic dispersion the solitons also have an asymmetric spectrum [40], which, based on physical reasons, is also expected to extend to any odd order of dispersion.

3.1. Energy and power scaling

It can be straightforwardly ascertained that if $u(T)e^{i\mu z}$ is a solution to Eq. (1), then so is $\alpha u(T/\alpha)e^{i\alpha^2\mu z}$, where α is real and positive. This scaling relation implies that, given a solution, then so is the scaled version with a temporal width that is α times smaller and an intensity that is α^2 times larger, and hence an energy that is α times larger. Indeed, for the soliton solution the energy is inversely proportional to the width or $E \propto \tau_0^{-1}$. As the pulse narrows in time, it broadens in frequency, strengthening the effect of dispersion, which needs to be balanced by an increased nonlinear effect, i.e., by an increased intensity.

A similar argument shows that for pure-quartic solitons $E \propto \tau_0^{-3}$, and that for a soliton with pure dispersion of order M, $E \propto \tau_0^{-(M-1)}$. This implies that the larger the order of dispersion the faster the rate at which the pulse energy grows; this may have application in soliton-based lasers. With our colleagues we experimentally demonstrated these energy-width scaling laws up to $M = 10$ using a reconfigurable

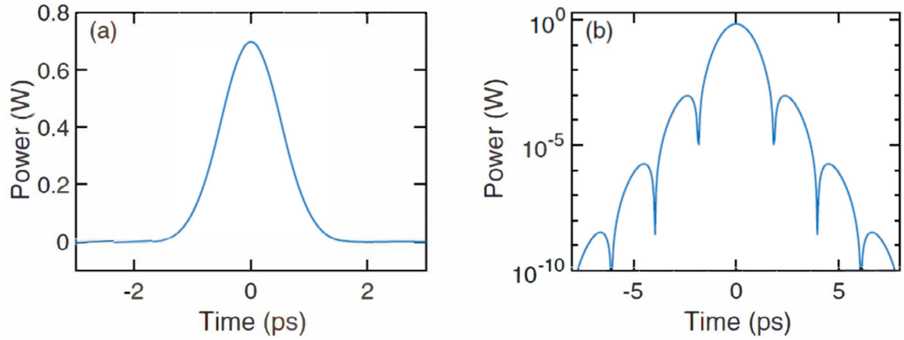

Fig. 1. Power versus time for a Pure Quartic Soliton with $\beta_4 = -2.2$ ps^{4}mm^{-1}, $\gamma = 4.072$ W^{-1}mm^{-1}, and $\mu = 1.76$ mm^{-1}. (a) linear scale and, (b) logarithmic scale. Figure taken from Tam et al. [18].

ultrafast cavity [19,34]. A direct interpretation of the energy-width scaling law $E_M \propto \tau_0^{-(M-1)}$ would imply that moving to high-order PHEODS would undoubtedly entail higher-energy pulses in the ultrashort pulse regime. However, a more nuanced look at the energy scaling shows that

$$E_M = K_M \frac{|\beta_M|}{\gamma \tau^{(M-1)}}, \tag{7}$$

with K_M an energy coefficient that decreases rapidly with M [19].[1] Therefore, the advantageous dependence with τ may be trumped by the smaller energy coefficient at large M. This, combined with the practical difficulty of implementing propagation media or cavities with dominant high-orders of dispersion, may impose a practical limit on the highest-order PHEODS of interest for high-energy ultrafast applications. Thus, the potential of PHEODS for generating high-power ultrashort pulses remain to be fully understood.

In the presence of mixed orders of dispersion, solutions to the generalized nonlinear Schrödinger equation Eq. (4) are also subject to scaling relations. However, a change in width of a solution changes the relative strengths of different orders of dispersion and in this case the scaling relation thus involves the scaling of dispersion coefficients as well.

3.2. Soliton shape

The nonlinear Schrödinger equation Eq. (1) has soliton solutions with a hyperbolic secant form [3,39,42] The solutions for higher, even orders of dispersion are not analytically known. The soliton solution in the presence of pure 4th order dispersion was discussed by Tam et al. [18] (see Fig. 1). While the detailed shape of the solution had to be determined numerically, some of the key properties, such as the shape of the low-intensity tails and energy and power scaling can be determined analytically.

At low intensities, such as occurs in the soliton tails, nonlinear effects can be neglected and Eq. (6) simplifies considerably. The electric field envelope can then be written as a linear superposition of terms of the type $e^{\eta T}$, where, for pure dispersion of order M, the η are the M roots of

$$\eta = (\pm 1)^{1/M} \left(M!\,\mu/|\beta_M|\right)^{1/M}. \tag{8}$$

The + sign applies when $M/2$ is odd, whereas - applies for $M/2$ even. Half of the roots, namely those with $\Re(\eta) > 0$, correspond to the leading edge of the pulse, whereas the other half, with $\Re(\eta) < 0$, correspond to its trailing edge.

For $M = 2$ the soliton tails vary as $e^{\pm\sqrt{2\mu/|\beta_2|}T}$ [3], whereas for $M = 4$ they vary as $e^{\pm|\eta T|/\sqrt{2}} \cos(|\eta T|/\sqrt{2}+\alpha_4)$ where $|\eta| = (24\mu/|\beta_4|)^{1/4}$, and where we used that u is real. The oscillations in the tail are illustrated in Fig. 1(b). The phase α_4 cannot be determined by this method and requires solving the full nonlinear equation.

The argument for $M > 4$ requires one additional step. In the presence of pure M^{th} order dispersion there are M distinct values for η, all with magnitude $|\eta| = (M!\mu/|\beta_M|)^{1/M}$ (see Eq. (8)). Their arguments are shown in Fig. 2. Fig. 2(a) shows the result for $M/2$ is odd ($M = 10$), whereas Fig. 2(b) shows the result for $M/2$ even ($M = 12$). The roots are never imaginary, which would lead to solutions that are of infinite extent. The most relevant of these roots are the ones with the smallest magnitude of their real part since they decay most slowly away from the central peak. These are the four roots with the arguments $\vartheta_M = \pm(1/2 \pm 1/M)\pi$ (indicated by squares in Fig. 2). The leading and trailing edges of the solitons thus have the functional form $e^{\pm|\eta T \cos(\vartheta_M)|} \cos(|\eta T \sin(\vartheta_M)| + \alpha_M)$. Since ϑ_M increases with M, the oscillations grow denser as M increases.

As mentioned in Section 3.1 the intrinsic soliton shape for pure, higher-order even dispersion is independent of μ and of β_4. In the presence of mixed dispersion, the soliton shape does depend on the peak power though the parameter μ. As a general rule for small μ the low orders in the Taylor expansion dominate (typically $m = 2$), and the solution is similar to a hyperbolic secant. As μ increases, the solitons start to depend on higher dispersion orders, and evolve to have oscillating tails, as mentioned in the paragraphs above. Eventually, for the largest μ, the highest order dispersion term ($m = M$) dominates. Thus, as μ increases, and the pulse becomes more nonlinear, the oscillations in the tails become increasingly dense.

3.3. Galilean invariance

The soliton solutions to the nonlinear Schrödinger equation have Galilean invariance. This means that their shape is independent of their velocity—in other words, the equation is independent of the frame in which it is written. The reason is that the linear dispersion relation is taken to be parabolic, so that the local Taylor expansion Eq. (3), truncated after the quadratic term, is the same in any co-moving frame.

For pure, higher-order dispersion relations this is no longer true. For example, a quartic dispersion relation has a unique frequency where $\beta_2 = 0$ and $\beta_3 = 0$ simultaneously. The dispersion relation in the frame corresponding to the group velocity β_1^{-1} at this frequency is a pure quartic polynomial. However, in any other frame this is no longer true—the quartic term, when expanded at some other frequency generates quadratic and cubic terms [40]. This means that pure quartic solitons can propagate only at a unique speed. At other speeds, the emergence of β_3 (in addition to the emergency of β_2), causes the solitons to have an asymmetric spectrum, though they remain symmetric in time. The presence of the β_3 term, or any odd order of dispersion (see Appendix), also cause the solitons to have a non-trivial chirp, unlike solitons with only even orders of dispersion [40].

[1] The decrease in size of the K_M with M is related to the expressions for the effective dispersion lengths discussed in Section 3.5

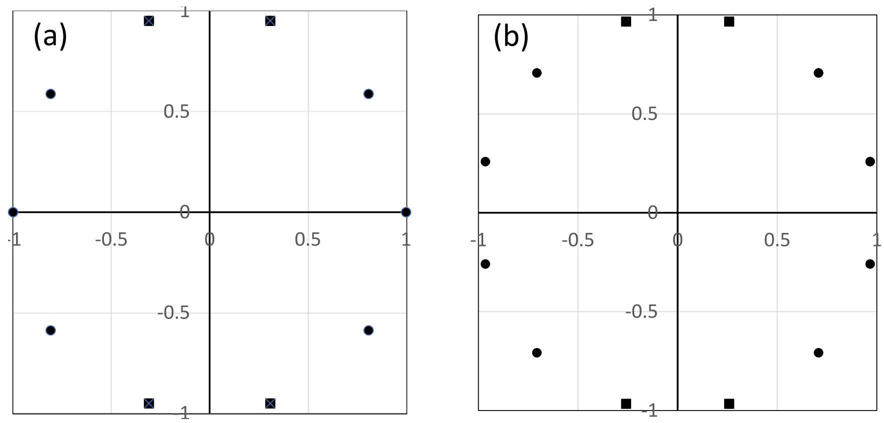

Fig. 2. Arguments of the roots for (a): $M = 10$, and (b) $M = 12$. The roots with the smallest magnitudes of their real parts are indicated by squares, whereas the other roots are indicated by circles.

3.4. Nonlinear and dispersive phase compensation

In an instructive 2001 paper, Dudley, Peacock and Millot introduced an intuitive physical explanation of how the nonlinear and dispersive phase shifts compensate each other to form an NLS soliton [43]. They aimed to debunk the common misconception that the dispersive phase shift is necessarily parabolic and, therefore, the dispersive frequency gradient (the chirp), is always linear. With this misconception in mind it is hard to envision how the dispersive phase shift can compensate the intensity dependent nonlinear phase shift. Dudley et al. showed that while at long propagation distances the dispersive phase shift is parabolic and the chirp is thus linear, at short propagation distances this is not the case, and indeed it can perfectly compensate for the nonlinear phase shift across the entire pulse. More recently, we applied this chirp compensation argument to justify the approximate Gaussian shape of pure-quartic solitons around the center of the pulse [17]. Here, equipped with exact numerical solutions for PHEODS with various orders of dispersion, we generalize this argument to the entire PHEODS hierarchy with the purpose of highlighting the similar physical origin of all the members of this soliton family but also some important practical differences.

As input condition to our simulations we use the temporal amplitudes of the pure-quartic soliton ($M = 4$), pure-sextic soliton ($M = 6$), and pure-octic soliton ($M = 8$) are shown in Figs. 3(a), (f), and (k), respectively. These are numerically obtained assuming stationary solutions to the NLSE with $\gamma = 1$ W^{-1}m^{-1} and $\beta_4 = -0.7853$ ps^4/m, $\beta_6 = -0.6167$ ps^6/m, and $\beta_8 = -0.4842$ ps^8/m, respectively, and solving the resulting ODE with the Newton-conjugate gradient method [18,44]. The resulting pure-quartic, pure-sextic, and pure-octic solitons have full width at half maximum pulse durations of 0.9 ps, 0.75 ps, and 0.63 ps, respectively. While interesting oscillatory dynamics occur in the tails of these solitons (see Section 3.2), here we only show the center part of the solitons to avoid distractions from the main argument. We consider the propagation of these $M = 4, 6, 8$ PHEODs in an optical medium with the nonlinear and dispersive parameters described above by solving Eq. (4) numerically using the split-step Fourier method [3]. The nonlinear (solid green) and dispersive chirps (dash-dot blue) at increasingly small propagation distances are shown in Figs. 3 (b)–(e), (g)–(j) and (l)–(o) for $M = 4, 6, 8$ respectively. In order to present a fair comparison between the propagation lengths, agnostic to the dispersion order, we normalize the propagation length to the effective dispersive lengths L_M (see Section 3.5). For $M = 4$ and $M = 6$ we use $L_M = \tau^M/(N_M|\beta_M|)$, with τ the full width at half maximum pulse duration and N_M a numerically obtained prefactor derived in [19]. For $M = 8$ we use an analytic expression derived in [35] and discussed in Section 3.5.

The first observation is that, at relatively long propagation distances $L = L_M$, the chirp for all orders of dispersion is approximately linear and does not fully balance the nonlinear chirp across the pulse duration (see Figs. 3(b), (g) and (l)). As we decrease the propagation length to $L = L_M/10$, we observe how the quartic dispersion chirp starts to acquire a similar but opposite sign profile to the nonlinear chirp (Fig. 3(c)). However the sextic and octic dispersion chirps are still very much linear, as shown in Figs. 3(h) and (m). As the propagation length becomes $L = L_M/20$ the quartic dispersion chirp fully negates the nonlinear one (Fig. 3(d)), the sextic dispersion chirp starts to resemble the opposite profile to the nonlinear one (Fig. 3(i)), but the octic-dispersion chirp is still linear (Fig. 3(n)). At $L = L_M/100$, both the sextic and the octic dispersion chirps compensate virtually perfectly for the nonlinear chirp, as shown in Fig. 3(j) and (o) respectively.

These results provided evidence that the general statement that for *sufficiently short* propagation lengths the nonlinear and dispersive chirps compensate each other remains valid for all PHEODS. However, for increasing orders of dispersion, such *sufficiently short* propagation length becomes shorter and shorter. This can be understood by noting that the dispersive phase shift for the pure orders of dispersion is given by $\psi(z,\omega) = z\beta(\omega) = z\frac{\beta_m}{M!}\omega^M$. For sufficiently large bandwidths, the phase evolution with z occurs more rapidly for large M values.

Understanding the physics of phase compensation in PHEODS is interesting from a pedagogical perspective but it may also have practical implications in the context of soliton formation in ultrafast laser cavities with dominant high-order dispersions [19,34].

3.5. Effective lengths

Thus far, our statement that solitons can be thought of as balancing the effects of dispersion and nonlinearity was qualitative. To do this in a more quantitative way, we adopt the approach of Agrawal [3]. He introduces characteristic length scales associated with each physical effect occurring in a fiber or waveguide—the effect with the shortest characteristic length are then taken to dominate. Solitons can be considered to form when the nonlinear length L_{NL} equals the dispersion length L_M where M is the order of the dispersion. The nonlinear length takes the form [3]

$$L_{\mathrm{NL}} = \frac{1}{\gamma P}, \tag{9}$$

where P is the pulse's peak power. The dispersion for second order dispersion L_2 is well known [3]

$$L_2 = \frac{\tau_0^2}{|\beta_2|}, \tag{10}$$

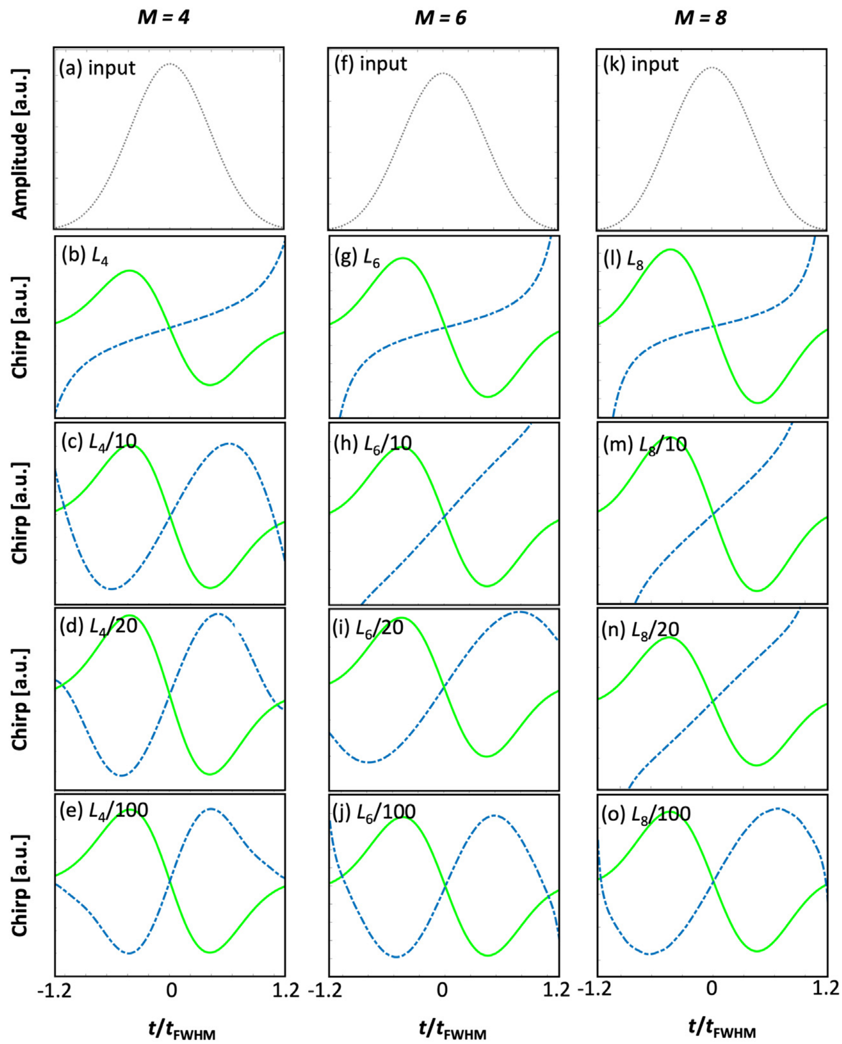

Fig. 3. Compensation of nonlinear and dispersive chirps for various high, even dispersion orders. (a)–(e) Amplitude of the pure-quartic soliton ($M = 4$) and the nonlinear (green) and dispersive phase shifts (dash-dot blue) for increasingly small propagation lengths of (b) L_4; (c) $L_4/10$; (d) $L_4/20$; (e) $L_4/100$. (f)–(j) Same for $M = 6$, and (k)–(o) for $M = 8$.

where τ_0 is a measure of the pulse length. In fact, when τ_0 is defined to be the time over which the pulse intensity decreases from the peak value P to P/e^2, then the equality $L_2 = L_{\rm NL}$ is the condition for a soliton.

The generalization of Eq. (10) to higher orders of dispersion is not obvious. One generalization is $L_M = \tau_0^M/|\beta_M|$, and indeed this was adopted by some authors, including ourselves [3,17,36,45,46]. However, there is no *a priori* justification of this, and we will see below that this leads to some unphysical consequences. Rather, one of us (CMdS) and co-workers argued that when M is sufficiently large ($M \gtrapprox 8$) [35]

$$L_M = M!\,\ell_c^M \frac{\tau_0^M}{|\beta_M|}, \tag{11}$$

where the dimensionless quantity $\ell_c = 0.332$, was found by comparing with numerical results, and τ_0 is the full-width at half-maximum of the pulse. The condition $L_M = L_{NL}$ then gives the condition for soliton formation. The prefactor grows faster than exponentially with M, indicating that simply generalizing the result in Eq. (10) strongly overestimates the effect of the dispersion for large M.

Let us consider one of the consequences of Eq. (11). Since the phase φ upon propagation over a length L is $\varphi = \beta L$, we find with Eq. (3) that

$$\varphi_M(\omega) = \frac{\beta_M}{M!}(\omega-\omega_0)^M L, \tag{12}$$

if dispersion of order m dominates. With Eq. (11) this can be rewritten as

$$\varphi_M(\omega) = (\ell_c \tau_0(\omega-\omega_0))^M \frac{L}{L_M}. \tag{13}$$

For large M and for fixed L this function is initially very small and then increases rapidly once the right-hand side exceeds unity. Thus, for frequencies such that $|\omega-\omega_0|$ is small enough that $\varphi_M \lessapprox 1$, the effect of the dispersion is negligible.

The right-hand side of Eq. (13) takes the value of unity when $(\omega-\omega_0)\tau_0\ell_c = (L_M/L)^{1/M}$. Now for most conventional pulses the time–bandwidth product in terms of angular frequency is approximately 2 or 3 (it is 1.98 for hyperbolic secant squared pulses and 2.76 for Gaussian pulses). Therefore at $L = L_M$, and since $\ell_c = 0.332$, the fraction of the pulse that is still unaffected by the dispersion has a bandwidth roughly corresponding to the full-width at half maximum of the pulse spectrum. This fraction decreases slowly with propagation because of the $1/M$ power in $(L_M/L)^{1/M}$. The remainder of the pulse has a group velocity that differs strongly from that at ω_0, and forms a pedestal in the time domain. We note that Eq. (11) is only valid for large M, in practice $M \gtrapprox 8$, and so it would not be expected to apply for $M = 2$. Nonetheless, the prefactor in this case is 0.664 which is close to unity.

The alternative definition in which L_M equals $\tau_0^M/|\beta_M|$, i.e., without the prefactor in Eq. (11), would lead to the conclusion that the frequency range that is unaffected by the dispersion equals $(\omega-\omega_0)\tau_0 =$

$(M!)^{1/M} \approx M/e$, where we used the Stirling approximation [47] in the last step. This would lead to the misleading conclusion that for large M, at L_M a negligible fraction of the pulse is affected by the dispersion.

4. Discussion & conclusions

We have provided pedagogical arguments for the formation of PHEODS based on the spectral variation of the inverse group velocity and phase compensation, which remain fundamentally the same as those for the formation of NLS. This is not surprising, as NLS and PHEODS are part of the same infinite hierarchy of solitons. We have also highlighted important differences in terms of energy scaling, effective lengths, and Galilean invariance. However, we have not discussed the fact the nonlinear Schrödinger equation Eq. (1) is integrable [48], whereas the generalized nonlinear Schrödinger equations Eq. (4) we have considered are not. We have therefore used the term "soliton" in the sense that it refers to stationary, pulse-like solutions in media with dispersion and a nonlinearity. We do not imply that these solutions are unaffected by collisions or any of the other characteristics of solutions of integrable systems. What we do imply is that from a practical perspective pure-quartic solitons and, more generally, PHEODS hold virtually all of the practical properties that make solitons useful and that some aspects, such as energy-width scaling and spectral flatness, they can entail advantages for applications in ultrafast lasers [34] and frequency combs [28].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix

In this appendix we initially consider the expression for the energy flow J in the presence of a pure order of dispersion. According to the method developed by Widjaja et al. [40], the energy flow satisfies

$$\frac{\partial J}{\partial T} \propto \begin{cases} \mathfrak{Im}\{\psi^\star \frac{\partial^M \psi}{\partial T^M}\} & M \text{ even,} \\ \mathfrak{Re}\{\psi^\star \frac{\partial^M \psi}{\partial T^M}\} & M \text{ odd.} \end{cases} \tag{14}$$

For large M the expression for J can be very complicated. However, the expression for $\psi^\star(\partial^M\psi/\partial T^M)$ always has the real term $|\psi|(\partial^M|\psi|/\partial T^M)$, where $|\psi|$ is the modulus of ψ. According to Eq. (14), for even M this term does not contribute to the energy flow. However for odd M it does, in which case we can write

$$|\psi|\frac{\partial^M|\psi|}{\partial T^M} = \frac{\partial}{\partial T}\left[|\psi|\frac{\partial^{M-1}|\psi|}{\partial T^{M-1}} - \frac{\partial|\psi|}{\partial T}\frac{\partial^{M-2}|\psi|}{\partial T^{M-2}} + \frac{\partial^2|\psi|}{\partial T^2}\frac{\partial^{M-3}|\psi|}{\partial T^{M-3}} - \cdots \right.$$
$$\left. - \frac{\partial^{(M+1)/2}|\psi|}{\partial T^{(M+1)/2}}\frac{\partial^{(M-3)/2}|\psi|}{\partial T^{(M-3)/2}} + \frac{1}{2}\left(\frac{\partial^{(M-1)/2}|\psi|}{\partial T^{(M-1)/2}}\right)^2\right]. \tag{15}$$

This implies that in the presence of at least one odd order of dispersion, the expression for the energy flow contains at least one term that only depends on the modulus of ψ and not on its argument. Such terms thus contribute to the energy flow even though the pulse may be unchirped. Hence, in the presence of odd dispersion, it is not sufficient for a pulse not to be chirped to be stationary.

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