Reflection of light by composite volume holograms: Fresnel corrections and Fabry–Perot spectral filtering

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Effects in composite volume Bragg gratings (VBGs) are studied theoretically and experimentally. The mathematics of reflection is formulated with a unified account of Fresnel reflections by the boundaries and of VBG reflection. We introduce the strength S of reflection by an arbitrary lossless element such that the intensity of reflection is $R = \tanh^2 S$. We show that the ultimate maximum/minimum of reflection by a composite lossless system corresponds to addition/subtraction of relevant strengths of the sequential elements. We present a new physical interpretation of standard Fresnel reflection: Strength for TE or for TM reflection is given by addition or by subtraction of two contributions. One of them is an angle-independent contribution of the impedance step, while the other is an angle-dependent contribution of the step of propagation speed. We study an assembly of two VBG mirrors with a thin immersion layer between them that constitutes a Fabry–Perot spectral filter. The transmission wavelength of the assembly depends on the phase shift between the two VBGs. Spectral resolution $\Delta\lambda(\text{FWHM}) = 25$ pm at $\lambda = 1063.4$ nm is achieved with the device of small total physical thickness $2L = 5.52$ mm.

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1. INTRODUCTION

Propagation of light in layered media is one of the fundamental problems in general electrodynamics and in optics in particular. Almost any textbook devotes considerable space to this problem, where, by definition of the term "layered," all the media properties depend on one coordinate only (for definiteness, on z); see, e.g. [1–3]. Specialized texts devoted to layered media are abundant as well; see, e.g. [4,5]. At the same time, diffractive transformations of light beams by volume Bragg gratings (VBGs) has been considered in numerous publications; see, e.g. [6–8].

One of the purposes of the present paper is to consider the interference between three processes of reflection. The first and third ones are the Fresnel reflections by the boundaries of a parallel glass plate. The second is the process of holographic reflection by a VBG located inside that glass plate; see Fig. 1.

There is a certain technical difficulty in the simultaneous consideration of all three processes. Fresnel reflections require "stitching" full electromagnetic fields via boundary conditions, and the step in the refractive index may be arbitrarily large there. On the other hand, reflection by a VBG is correctly considered in the slowly varying envelope approximation (SVEA), since the modulation n_1 of the refractive index profile $n(z)$,

$$n(z) = n_0 + n_2(z) + n_1'(z)\cos[Qz + \gamma(z)] + in_1''(z)\cos[Qz + \delta(z)], \tag{1.1}$$

is assumed to be small: $|n_1| \ll n_0$. Indeed, for the VBGs of our interest typically $|n_1| \le 3 \times 10^{-4} n_0$; see [9,10]. Equation (1.1) also takes into account possible deviation $n_2(z) = n_2' + in_2''$ of a smooth refractive index profile from its constant real value n_0. The imaginary part of $n_2(z)$ takes into account background absorption. The grating in Eq. (1.1) in most of this paper is assumed to be purely refractive, $n_1(z) \equiv \text{Re}[n_1(z)]$, but the basic equations will allow for $in_1''(z)$ as well.

To overcome these differences in description of discrete boundary reflection and VBG reflection, we reformulate the approach to the solutions of Maxwell's equations. Namely, we produce an exact system of first-order coupled equations for the counterpropagating waves. That system allows us to account, in a unified manner, for both SVEA and reflection by sharp steps. It is worth noting that the same task could, in principle, be performed by existing matrix methods. The approach closest to ours is adopted in the text [5]; nevertheless, there are some particular important differences.

An important role in our approach is assigned to the parameter S, *strength of reflection* by a lossless element: S is defined in such a way that the intensity reflection coefficient may be written as

$$R = |r|^2 = \tanh^2 S. \tag{1.2}$$

An extra advantage of introduction of the strength parameter S is that even if $S \to \infty$, reflection intensity $|r|^2$ approaches 1 only asymptotically for media without loss or gain.

Our consideration of the general electrodynamic problem takes into account two distinctly different mecha-

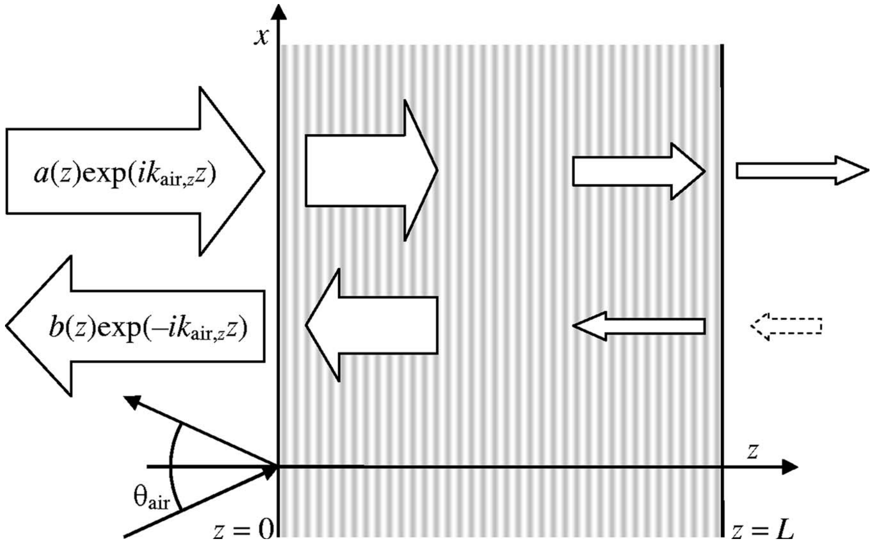

Fig. 1. Notations for the incident $a(z)$ and reflected $b(z)$ waves in the approximation of infinitely wide plane beams with account of the reflections from both boundaries, $z=0$ and $z=L$, as well as of the reflection by a VBG.

nisms of reflection: by the gradients of propagation speed v, and by the gradients of impedance Z. Here ε and μ are the local values of dielectric permittivity and magnetic permeability, respectively, while ε_{vac} and μ_{vac} are the respective values in vacuum, where the speed of light $c=1/\sqrt{\varepsilon_{\text{vac}}\mu_{\text{vac}}}=3\times 10^8$ m/s, n is local refractive index, and

$$v=\frac{c}{n}, \quad Z=\sqrt{\frac{\mu}{\varepsilon}}, \quad n=\sqrt{\frac{\varepsilon\mu}{\varepsilon_{\text{vac}}\mu_{\text{vac}}}}. \tag{1.3}$$

We were able to show that the strength S of the reflection from a single sharp boundary is a linear sum (for TE polarization) or a linear difference (for TM polarization) of two contributions. One of these contributions does not depend on polarization and on impedance, but is governed by the change of propagation direction: a kinematic effect depending only on the step in the refractive index (i.e., of propagation speed). The other contribution also does not depend on polarization; neither does it depend on the incidence angle. This contribution is provided only by the step in the impedance Z. In this sense in the present paper we claim new understanding of old phenomena: Fresnel reflection and the Brewster effect.

We use a particular kind of matrix approach to the waves in layered media (compare also with the approach from [5]). While the incident and reflected waves are propagating in ($+z$) and ($-z$) directions, respectively, our approach requires the solution of the system of coupled ordinary differential equations (ODE) in one direction only (e.g., $+z$). In this way we reduce the original two-boundary ($z=0$ and $z=L$) problem to the Cauchy problem of an ODE system, i.e., to the one with initial conditions at $z=0$ only. That allows use of various analytical and/or numerical methods of solving these ODE.

The results of our consideration of the mutual influence of VBG reflections and Fresnel reflections depend explicitly on their relative phases. An important general conclusion is that the argument S of reflection function $R=\tanh^2 S$ by a lossless element lies within the limits

$$S_{\text{VBG}}-(|S_1|+|S_2|)\le S\le S_{\text{VBG}}+(|S_1|+|S_2|), \tag{1.4}$$

where S_{VBG}, S_1, and S_2 are the strength parameters for VBG and for the two boundaries, respectively.

We consider a combined VBG that has a shift between the phases of two gratings. Actually, in recent years, phase-shifted fiber Bragg gratings have drawn great interest by researchers [11,12]. Under introduction of a π phase shift into a fiber Bragg grating during its fabrication, the spectral transmission acquires a narrow bandpass appearing within the middle of the central lobe of the fiber Bragg grating. In [9,10], the authors established how to record high efficiency and low loss VBG in photo-thermorefractive (PTR) glass. In using this material a relative diffraction efficiency as high as 99.9% has been achieved, and the level of losses has been kept below 10^{-2} cm^{-1}. Such material therefore represents an ideal one for the fabrication of a combined VBG filter. In Section 8 we present analytical and numerical calculations of such a cavity, and compare them with the solutions of similar problems for distributed feedback and related systems [13–16]. Finally we present an experimental demonstration of this filter in air.

2. TRANSFORMATIONS OF MAXWELL EQUATIONS

It is instructive to write the Maxwell equations for the general case, when both dielectric permittivity $\varepsilon(z)$ [farad/m] and magnetic permeability $\mu(z)$ [henry/m] are functions of z. Later we will limit ourselves to the optical case, where $\mu=\mu_{\text{vac}}$, $\varepsilon(z)=\varepsilon_{\text{vac}}n^2(z)$. We take the Maxwell equations for electric field vector $\mathbf{E}(\mathbf{r},t)$ [volt/m] and magnetic vector $\mathbf{H}(\mathbf{r},t)$ [ampere/m] as

$$\varepsilon(\mathbf{r})\frac{\partial\mathbf{E}}{\partial t}=\nabla\times\mathbf{H}, \quad \mu(\mathbf{r})\frac{\partial\mathbf{H}}{\partial t}=-\nabla\times\mathbf{E}. \tag{2.1}$$

Maxwell's equations (2.1) may be written in a form that explicitly shows certain symmetry between $\mathbf{E}$ and $\mathbf{H}$:

$$\mathbf{H}(\mathbf{r},t) = \mathbf{h}(\mathbf{r},t)\frac{1}{\sqrt{Z(\mathbf{r})}}, \quad \mathbf{E}(\mathbf{r},t) = \mathbf{e}(\mathbf{r},t)\sqrt{Z(\mathbf{r})},$$

$$\mathbf{g} = -\tfrac{1}{2}\nabla \ln Z(\mathbf{r}), \tag{2.2}$$

$$\frac{n}{c}\frac{\partial \mathbf{h}}{\partial t} = -\nabla \times \mathbf{e} + \mathbf{g} \times \mathbf{e}, \quad \frac{n}{c}\frac{\partial \mathbf{e}}{\partial t} = \nabla \times \mathbf{h} + \mathbf{g} \times \mathbf{h}, \tag{2.3}$$

Below we will use the monochromatic amplitudes of the type

$$\mathbf{A}_{\mathrm{real}}(\mathbf{r},t) = \tfrac{1}{2}[\mathbf{A}(\mathbf{r})\exp(-i\omega t) + \mathbf{A}^*(\mathbf{r})\exp(i\omega t)], \tag{2.4}$$

where $\omega = 2\pi c/\lambda_{\mathrm{vac}}$, and c is speed of light in vacuum. Advantages of the complex amplitudes $\mathbf{e}$ and $\mathbf{h}$ are as follows: (1) For a plane wave propagating through the medium with constant real refractive index n and constant real impedance Z in direction $\mathbf{m} = \mathbf{k}/|\mathbf{k}|$, one has $|\mathbf{e}| = |\mathbf{h}|$, $\mathbf{e} = -[\mathbf{m} \times \mathbf{h}]$, $\mathbf{h} = [\mathbf{m} \times \mathbf{e}]$; (2) the time-averaged Poynting vector $\mathbf{S}$ [watt/m^2] in a region of the medium where the loss is absent may be expressed via $\mathbf{e}$ and $\mathbf{h}$ as $\mathbf{S} = \langle \mathbf{e}_{\mathrm{real}} \times \mathbf{h}_{\mathrm{real}} \rangle_t = (\mathbf{e} \times \mathbf{h}^* + \mathbf{e}^* \times \mathbf{h})/4$.

We consider the incidence plane to be the (x,z) plane for a wave of λ_{vac} incident on a layer $0 \le z \le L$ of the medium, with the properties being z dependent only. By θ_{air} we denote the incidence angle of the wave in air, so that

$$\mathbf{k}_{\mathrm{air}} = \hat{\mathbf{x}} k_x + \hat{\mathbf{z}} k_{\mathrm{air},z}, \quad k_x = \frac{\omega}{c} n_{\mathrm{air}} \sin \theta_{\mathrm{air}},$$

$$k_{\mathrm{air},z} = \frac{\omega}{c} n_{\mathrm{air}} \cos \theta_{\mathrm{air}}. \tag{2.5}$$

In most cases the approximation $n_{\mathrm{air}} \approx 1$ works quite well. Nevertheless, we keep it for the case of large θ_{air}, when high angular selectivity of the VBG may require a more accurate value of n_{air}. The incidence of light on a hologram's surface from some other medium than air may also be covered by Eq. (2.5), by assigning an appropriate value to n_{air}. Incidence angle θ_{air} means x-dependence of the wave amplitudes proportional to $\exp(ik_x x)$, and k_x is the same at all z. The waves in a layered medium are naturally separated into transverse electric (TE) and transverse magnetic (TM) parts, with components u_x, u_y, u_z for the electric vector of both TE and TM waves and w_x, w_y, w_z for the magnetic vector of these waves:

$$\text{TE:} \quad \mathbf{e}(\mathbf{r}) = -\hat{\mathbf{y}} u_y(z)\exp(ik_x x),$$

$$\mathbf{h}(\mathbf{r}) = [\hat{\mathbf{x}} w_x(z) + \hat{\mathbf{z}} w_z(z)]\exp(ik_x x); \tag{2.6}$$

$$\text{TM:} \quad \mathbf{e}(\mathbf{r}) = [\hat{\mathbf{x}} u_x(z) + \hat{\mathbf{z}} u_z(z)]\exp(ik_x x),$$

$$\mathbf{h}(\mathbf{r}) = \hat{\mathbf{y}} w_y(z)\exp(ik_x x). \tag{2.7}$$

Here and below we use the quantities $k(z)$, $p(z)$, $g(z)$, $f(z)$ defined by

$$k(z) = \frac{\omega}{c} n(z), \quad p(z) = \sqrt{k^2(z) - k_x^2},$$

$$g(z) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z} \ln Z(z), \quad f(z) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{p(z)}{k(z)}. \tag{2.8}$$

Maxwell's equations in the form (2.3) for the monochromatic amplitudes u_y, w_x, w_z in case of TE polarization are

$$iku_y = \partial_z w_x - ik_x w_z + g w_x, \quad -ikw_x = -\partial_z u_y + g u_y,$$
$$-ikw_z = ik_x u_y. \tag{2.9}$$

The system of Eqs. (2.9) may be rewritten for the two functions $u_y(z)$ and $w_x(z)$ only:

$$\partial_z u_y = g u_y + ikw_x, \quad \partial_z w_x = ip^2/k u_y - g w_x. \tag{2.10}$$

It is convenient to express the functions $u_y(z)$ and $w_x(z)$ via newly introduced amplitudes $a(z)$ and $b(z)$ for TE polarization:

$$a_{\mathrm{TE}}(z) = \frac{1}{\sqrt{8}}\left[\sqrt{\frac{p}{k}} u_y(z) + \sqrt{\frac{k}{p}} w_x(z)\right] e^{-ik_{\mathrm{air},z} z},$$

$$b_{\mathrm{TE}}(z) = \frac{1}{\sqrt{8}}\left[\sqrt{\frac{p}{k}} u_y(z) - \sqrt{\frac{k}{p}} w_x(z)\right] e^{ik_{\mathrm{air},z} z}. \tag{2.11}$$

The physical sense of the amplitudes $a_{\mathrm{TE}}(z)$ and $b_{\mathrm{TE}}(z)$ is especially clear when the propagation direction of the incident wave is close to the z axis. Then

$$a_{\mathrm{TE}}(z) \approx -\frac{E_y}{\sqrt{8Z}} + \sqrt{\frac{Z}{8}} H_x, \quad b_{\mathrm{TE}}(z) \approx -\frac{E_y}{\sqrt{8Z}} - \sqrt{\frac{Z}{8}} H_x. \tag{2.12}$$

The Poynting vector's z component (for any incidence angle in a lossless part of the medium) according to Eqs. (2.6) and (2.11) is

$$S_z = |a_{\mathrm{TE}}|^2 - |b_{\mathrm{TE}}|^2. \tag{2.13}$$

If the medium exhibits weak loss, then Eq. (2.13) is still valid with good accuracy. In the regions where the impedance $Z(z) = \mathrm{const}_1$, and the refractive index has real value $n(z) = \mathrm{const}_2$, the moduli of our amplitudes $|a_{\mathrm{TE}}|$ and $|b_{\mathrm{TE}}|$ stay constant. The amplitudes and phases of a_{TE} and b_{TE} stay constant under propagation in "air." The exact Maxwell Eqs. (2.10) for TE polarization are reduced to the simple coupled pair

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix} a_{\mathrm{TE}}(z) \\ b_{\mathrm{TE}}(z) \end{bmatrix} = \hat{V}_{\mathrm{TE}} \begin{bmatrix} a_{\mathrm{TE}}(z) \\ b_{\mathrm{TE}}(z) \end{bmatrix},$$

$$\hat{V}_{\mathrm{TE}} = \begin{bmatrix} i[p(z) - k_{\mathrm{air},z}] & [g(z) + f(z)]e^{-2ik_{\mathrm{air},z} z} \\ [g(z) + f(z)]e^{2ik_{\mathrm{air},z} z} & -i[p(z) - k_{\mathrm{air},z}] \end{bmatrix}. \tag{2.14}$$

Similar transformations may be made for the field components (2.7) of the TM polarization:

$$-iku_x = -\partial_z w_y - g w_y, \quad -iku_z = ik_x w_y, -ikw_y = ik_x u_z - \partial_z u_x + g u_x \quad \Leftrightarrow \\ \Leftrightarrow \quad \partial_z u_x = g u_x + ip^2/k, \quad \partial_z w_y = iku_x - g w_y, \tag{2.15}$$

with the same parameters $k(z)$, $g(z)$, $p(z)$.

We now introduce the amplitudes of coupled $(+z)$ and $(-z)$ propagating waves for TM polarization by

$$a_{\mathrm{TM}}(z) = \frac{1}{\sqrt{8}}\left[\sqrt{\frac{p}{k}} w_y(z) + \sqrt{\frac{k}{p}} u_x(z)\right] e^{-ik_{\mathrm{air},z} z},$$

$$b_{\mathrm{TM}}(z) = \frac{1}{\sqrt{8}}\left[-\sqrt{\frac{p}{k}} w_y(z) + \sqrt{\frac{k}{p}} u_x(z)\right] e^{ik_{\mathrm{air},z} z}. \tag{2.16}$$

Finally, the exact Maxwell equations for TM polarization are

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix} a_{\mathrm{TM}}(z) \\ b_{\mathrm{TM}}(z) \end{bmatrix} = \hat{V}_{\mathrm{TM}} \begin{bmatrix} a_{\mathrm{TM}}(z) \\ b_{\mathrm{TM}}(z) \end{bmatrix},$$

$$\hat{V}_{\mathrm{TM}} = \begin{bmatrix} i[p(z) - k_{\mathrm{air},z}] & [g(z) - f(z)] e^{-2ik_{\mathrm{air},z} z} \\ [g(z) - f(z)] e^{2ik_{\mathrm{air},z} z} & -i[p(z) - k_{\mathrm{air},z}] \end{bmatrix}. \tag{2.17}$$

The expression for the z component of the Poynting vector is perfectly analogous to Eq. (2.13).

3. MATRIX APPROACH

Without coupling of the waves in a homogeneous medium our amplitudes a and b (be they of TE or TM polarization) would propagate according to the law

$$\begin{bmatrix} a(z_2) \\ b(z_2) \end{bmatrix} = \hat{K}((p - k_{\mathrm{air},z})(z_2 - z_1)) \begin{bmatrix} a(z_1) \\ b(z_1) \end{bmatrix}. \tag{3.1}$$

Here we have introduced the notation for the matrix $\hat{K}(\varphi)$:

$$\hat{K}(\varphi) = \begin{bmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{bmatrix}. \tag{3.2}$$

Reflection of the wave a by a specimen of arbitrary layered medium $z_1 \le z \le z_2$ is described by two separate boundary conditions: $a(z_1)=1$ at $z=z_1$ and $b(z_2)=0$ at $z=z_2$. The quantities sought are $r=b(z_1)$, $t=a(z_2)$, the amplitude reflection and transmission coefficients, respectively. Here $|r|^2$ and $|t|^2$ are reflection and transmission coefficients, respectively, for the intensities.

The numerical solution of the problem for ODE with conditions at two separate boundaries may present serious difficulties, including possible divergence of iterative procedures. To get around these difficulties, we will use the *linearity* of our system of ODE, which allows us to write the relationships

$$\begin{bmatrix} a(z_2) \\ b(z_2) \end{bmatrix} = \hat{M}(z_2, z_1) \begin{bmatrix} a(z_1) \\ b(z_1) \end{bmatrix}, \quad \hat{M}(z_2, z_1) = \begin{bmatrix} M_{aa} & M_{ab} \\ M_{ba} & M_{bb} \end{bmatrix}. \tag{3.3}$$

Here the matrix $\hat{M}(z_2, z_1)$ satisfies the following equation and the initial condition, respectively:

$$\frac{\mathrm{d}\hat{M}(z, z_1)}{\mathrm{d}z} = \hat{V}(z)\hat{M}(z, z_1), \quad \hat{M}(z_1, z_1) = \hat{1}, \tag{3.4}$$

where the particular expressions for evolution matrix $\hat{V}(z)$ for TE and TM polarizations are given by Eqs. (2.14) and (2.17). The fact that the trace of the evolution matrix $\hat{V}$ is zero for both TE and TM polarizations leads to an important property of matrix $\hat{M}(z_2, z_1)$:

$$\det \hat{M} = 1. \tag{3.5}$$

Then application of the boundary conditions $a(z_1)=1$ and $b(z_2)=0$ allows finding r and t as

$$r(b \leftarrow a) = b(z_1) = -\frac{M_{ba}(z_2, z_1)}{M_{bb}(z_2, z_1)},$$

$$t(a \leftarrow a) = a(z_2) = \frac{\det \hat{M}(z_2, z_1)}{M_{bb}(z_2, z_1)} = \frac{1}{M_{bb}(z_2, z_1)}, \tag{3.6}$$

and we used Eq. (3.5). Similarly, applying boundary conditions $a(z_1)=0$ and $b(z_2)=1$, one gets transmission and reflection coefficients for the case when the incident wave is $b(z)$, i.e., comes from $z=+\infty$:

$$r(a \leftarrow b) = a(z_2) = +\frac{M_{ab}(z_2, z_1)}{M_{bb}(z_2, z_1)},$$

$$t(b \leftarrow b) \equiv t(a \leftarrow a) = b(z_1) = \frac{1}{M_{bb}(z_2, z_1)}. \tag{3.7}$$

The advantage of using the $\hat{M}$ matrix is the possibility of solving the standard Cauchy problem for our ODE, i.e., the problem with "initial" conditions for a and b. Another aspect of the usefulness of the $\hat{M}$ matrix is that one can calculate such matrices for each of the constituent elements of the device, and then just multiply them in order of increasing coordinate z. In some cases we will write matrix $\hat{M}(z)$ depending on one argument that will correspond to the initial coordinate $z_1=0$.

With the assumption of no loss or gain our system preserves the Poynting vector, $\partial S_z/\partial z=0$. In that case our system of equations satisfies the conservation law:

$$|a|^2 - |b|^2 = \text{const (no-loss assumption)}. \tag{3.8}$$

To explain the consequences of this important conservation law, let us discuss for a moment the case of transformation between waves a and b in a transmission VBG. A plus sign (instead of the minus sign just above) would be valid for transmission holograms: $|a|^2+|b|^2=\text{const}$ (no loss, transmission). We know that matrices of z evolution would satisfy the unitary condition, $\hat{U}^{*T}(z)\hat{U}(z)=1$ for transmission lossless holograms. Our minus sign means that the $\hat{M}(z)$ matrix for reflection by a lossless medium satisfies the condition

$$\hat{M}^{*T}\hat{\eta}\hat{M}=\hat{\eta},$$

$$\hat{\eta}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \Rightarrow |M_{aa}|^2-|M_{ba}|^2=1=|M_{bb}|^2-|M_{ab}|^2;$$

$$M_{ab}^{*}M_{aa}-M_{bb}^{*}M_{ba}=0. \tag{3.9}$$

Equation (3.9) and the unity determinant condition mean that matrix $\hat{M}$ belongs to the SU(1,1) group, i.e., it is a complex analog of Lorentz group SL(1,1). Indeed, (1) the unit matrix belongs to our set; (2) if a matrix belongs to our set, then the inverse matrix exists and belongs to that set as well; (3) the product of two matrices is associative and also belongs to our set. The most general form of such matrix $\hat{M}$ depends on three real parameters: strength S and phases α and β:

$$\hat{M}=\begin{bmatrix}e^{i\alpha} & 0\\ 0 & e^{-i\alpha}\end{bmatrix}\hat{\Sigma}(S)\begin{bmatrix}e^{-i\beta} & 0\\ 0 & e^{i\beta}\end{bmatrix}\equiv\hat{K}(\alpha)\hat{\Sigma}(S)\hat{K}(-\beta)$$
$$\equiv\begin{bmatrix}e^{i(\alpha-\beta)}\cosh S & e^{i(\alpha+\beta)}\sinh S\\ e^{-i(\alpha+\beta)}\sinh S & e^{i(\beta-\alpha)}\cosh S\end{bmatrix},$$

$$\hat{\Sigma}(S)=\begin{bmatrix}\cosh S & \sinh S\\ \sinh S & \cosh S\end{bmatrix}. \tag{3.10}$$

Thus $\hat{\Sigma}(S)$, which may be called "matrix of hyperbolic rotation" (compare with the terminology of Lorentz transformations), is another basic building block for the problem of lossless reflection, along with $\hat{K}(\alpha)$. As a result the scattering matrix $\hat{U}$ is

$$\begin{bmatrix}a(+\infty)\\ b(-\infty)\end{bmatrix}=\hat{U}\begin{bmatrix}a(-\infty)\\ b(+\infty)\end{bmatrix}, \quad \hat{U}=\begin{bmatrix}t(a\leftarrow a) & r(a\leftarrow b)\\ r(b\leftarrow a) & t(b\leftarrow b)\end{bmatrix}. \tag{3.11}$$

$$t(a\leftarrow a)=e^{i(\alpha-\beta)}\frac{1}{\cosh S}, \quad r(a\leftarrow b)=e^{2i\alpha}\tanh S,$$

$$r(b\leftarrow a)=-e^{-2i\beta}\tanh S, \quad t(b\leftarrow b)=t(a\leftarrow a). \tag{3.12}$$

The scattering matrix $\hat{U}$ of such reflection hologram is unitary for media without absorption or gain. Thus one can express the strength parameter S via reflection or transmission as

$$S=\operatorname{arctanh}\sqrt{R}, \quad T=1-R \text{ (no-loss assumption)}. \tag{3.13}$$

To be able to understand better the physical meaning of individual matrix factors in Eq. (3.10), consider a problem of integration when the refection is lossless and is localized at $z=z_1\equiv z_2$; then it is described by

$$\hat{M}=\hat{\Sigma}(S)=\begin{bmatrix}\cosh S & \sinh S\\ \sinh S & \cosh S\end{bmatrix}, \quad t=\frac{1}{\cosh S},$$

$$|r|=|\tanh S|, \quad |t|^2+|r|^2=1. \tag{3.14}$$

Equations (3.10)–(3.12) mean that the reflection by the most general composite lossless element may be presented as if it consisted of a phase plate element $\hat{K}(\beta)$, pure reflector (3.14), and one more phase element $\hat{K}(\alpha)$.

The matrix $\hat{M}$ is useful even if absorption is present, but in this case it does not satisfy Eqs. (3.8)–(3.10), and the notion of reflection strength S is no longer useful. However, matrix $\hat{M}$ still has unit determinant, even in the case of loss present, and Eqs. (3.6) and (3.7) still hold. In that case our matrix must satisfy the inequalities

$$\frac{\mathrm{d}S_z}{\mathrm{d}z}\le 0\to 1-|M_{aa}|^2+|M_{ba}|^2\ge 0, \quad |M_{bb}|^2-|M_{ab}|^2-1\ge 0,$$

$$|M_{ab}^{*}M_{aa}-M_{bb}^{*}M_{ba}|$$
$$\le\sqrt{(1-|M_{aa}|^2+|M_{ba}|^2)(|M_{bb}|^2-|M_{ab}|^2-1)},$$

$$\text{all at } z\ge 0. \tag{3.15}$$

The matrices satisfying relations (3.15) constitute a mathematical object "semigroup with identity element." Namely, the unit matrix is a member of the set. The product of two matrices with zero or positive loss is associative and also belongs to the set. Moreover, an inverse matrix exists (since $\det\hat{M}=1$), but it does not always describe an element with loss, and thus sometimes it may belong to the set and sometimes not.

For better understanding of the possible structure of a matrix $\hat{M}$ satisfying relations (3.15), let us consider one more particular case of solutions of Eqs. (2.14) and (2.17). Namely, consider the propagation of waves in a medium with a presence of loss and with the power attenuation coefficient of the material $\alpha_{\text{loss}}[1/\text{m}]=2\omega n_2''/c$. One should also take the factor $1/\cos\theta_{\text{in}}$ to account for the path $\Delta L=\Delta z/\cos\theta_{\text{in}}$ of the ray in the material. In such a case, and ignoring the interaction between the counterpropagating waves, one gets

$$\hat{V} = \begin{bmatrix} i(p' - k_{\mathrm{air},z}) - \dfrac{\alpha_{\mathrm{loss}}}{2\cos\theta} & 0 \\ 0 & \dfrac{\alpha_{\mathrm{loss}}}{2\cos\theta} - i(p' - k_{\mathrm{air},z}) \end{bmatrix},$$

$$\frac{\mathrm{d}\hat{M}}{\mathrm{d}z} = \hat{V}\hat{M}, \quad \hat{M}(z) = \hat{K}((p' - k_{\mathrm{air},z})z)\hat{L}(Y), \tag{3.16}$$

$$\hat{L}(Y) = \begin{bmatrix} e^{-Y} & 0 \\ 0 & e^{Y} \end{bmatrix}, \quad Y = \frac{\alpha_{\mathrm{loss}} z}{2\cos\theta}. \tag{3.17}$$

In other words, the matrix $\hat{L}(Y)$ describes a pure attenuator by the amplitude factor $\exp(-Y)$ with no energy exchange between counterpropagating waves and with no phase shift. Still, we were not able to find the most general expression of the matrix satisfying the conditions (3.15) of no gain and of possible loss.

The most dramatic new property that loss may introduce is that now the power reflection coefficients $R_+ = |r(a \leftarrow b)|^2$ and $R_- = |r(b \leftarrow a)|^2$ may be not equal to each other. This is especially evident in the case when a strongly reflecting mirror, $R_0 \approx 1$, is attached to one side of a strongly absorbing piece of glass, $T_0 \approx \exp(-2Y) \ll 1$. In that case one of the reflection coefficients is large, e.g., $R_+ \approx 1$, while the other is very small, $R_- \approx \exp(-4Y) \ll 1$. Reciprocity relationship $T_+ = T_-$ is valid even in the case of loss or gain presence. In some cases R_+ and R_- may still be equal, even in the presence of loss or gain. Appendix A to this paper contains some further examples.

Our matrix approach is quite close to the approach used in the monograph [5] by Yeh. Our designations of coordinates x and z are interchanged in comparison with those of [5]. A more important difference is that our variant of Maxwell's equations (2.2) and (2.3) allows us to account for variations both of dielectric permittivity ε and of magnetic permeability μ, while the approach in [5] accounts for the optical case only, where $\mu \equiv \mu_{\mathrm{vac}} = \mathrm{const}$. Finally, there is a quite important difference in our normalization of amplitudes a and b in such a manner that the expression for the z component of the Poynting vector in a transparent region of the medium is Eq. (2.13), i.e., just the difference $|a|^2 - |b|^2$ of square moduli of amplitudes a and b. In the approach by Yeh in [5] one should multiply these square moduli by the local value of $(cn\cos\theta)/2$. By itself this simple factor does not make particular calculations from [5] any more difficult than ours. However, the possibility of writing the conservation law (3.8) and its consequence (3.9), as well as the most general expressions (3.8)–(3.10) for lossless media, including the notion of reflection strength S introduced in this paper, constitutes an important (in our humble opinion) advantage of our approach. This advantage is even more prominent with respect to inequalities (3.15), which any matrix $\hat{M}$ must satisfy in the presence of loss.

4. SHARP STEP OF REFRACTIVE INDEX AND OF IMPEDANCE: UNDERSTANDING FRESNEL REFLECTION FORMULAS

Fresnel reflection coefficients for the sharp boundary at $z=0$ between two transparent media with the parameters (Z_1, n_1, θ_1) and (Z_2, n_2, θ_2) are well known (our choice of the sign of the TM reflection amplitude is opposite to that adopted by the "Nebraska convention"; see [17]):

$$r_{\mathrm{TE}} \equiv r(E_y \leftarrow E_y) = \frac{\cos\theta_1/Z_1 - \cos\theta_2/Z_2}{\cos\theta_1/Z_1 + \cos\theta_2 Z_2},$$

$$r_{\mathrm{TM}} \equiv r(E_x \leftarrow E_x) = -\frac{Z_1\cos\theta_1 - Z_2\cos\theta_2}{Z_1\cos\theta_1 + Z_2\cos\theta_2}. \tag{4.1}$$

They must be accompanied by the Snell law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{4.2}$$

Consider the following formulas for two exceptional cases. The first case corresponds to the media 1 and 2 having exactly the same impedances $Z_1 = Z_2$ but different propagation speeds, i.e., different refractive indices $n_1 \neq n_2$. In that case both reflection coefficients are equal to each other (up to the sign):

$$r_{\mathrm{TE}} \equiv r(E_y \leftarrow E_y) = -r_{\mathrm{TM}} \equiv -r(E_x \leftarrow E_x) = \frac{\cos\theta_1 - \cos\theta_2}{\cos\theta_1 + \cos\theta_2}$$

$$\equiv -\tanh S_{\Delta n}, \quad S_{\Delta n} = \frac{1}{2}\ln\left(\frac{\cos\theta_2}{\cos\theta_1}\right). \tag{4.3}$$

The angular position of the Brewster effect moves to $\theta_1 = \theta_2 = 0$ for the case $Z_1 = Z_2$: no reflection at normal incidence to the boundary between a pair of impedance-matched media.

The second case corresponds to media 1 and 2 having exactly the same propagation speeds $n_1 = n_2$, but different impedances $Z_1 \neq Z_2$. In that (very unusual) case one gets $\theta_1 = \theta_2$, i.e., the propagation direction does not change under transition from one medium to the other. As a result, TE and TM amplitude reflection coefficients for this second case (1) are equal to each other, including the sign, and (2) are completely independent of the incidence angle:

$$r_{\mathrm{TE}} \equiv r(E_y \leftarrow E_y) = r_{\mathrm{TM}} \equiv r(E_x \leftarrow E_x) = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$\equiv -\tanh S_{\Delta Z}, \quad S_{\Delta Z} = \frac{1}{2}\ln\left(\frac{Z_1}{Z_2}\right). \tag{4.4}$$

It is instructive to see these simple results from our matrix approach, since we will use that approach below to account for mutual influence of Fresnel and VBG reflections. Sharpness of the boundary placed at $z = z_0$ allows us to consider the phase factors to be constant. By the choice of z origin one can make the phase become zero. Then our systems of Eqs. (2.14) or (2.17) are reduced to

$$\frac{\mathrm{d}a}{\mathrm{d}z} = [g(z) \pm f(z)]b, \quad \frac{\mathrm{d}b}{\mathrm{d}z} = [g(z) \pm f(z)]a, \quad (+)\mathrm{TE}, \quad (-)\mathrm{TM}. \tag{4.5}$$

Direct substitution allows us to verify that the following functions constitute exact solutions of Eqs. (4.5):

$$\begin{bmatrix} a(z) \\ b(z) \end{bmatrix} = \begin{bmatrix} \cosh S(z) & \sinh S(z) \\ \sinh S(z) & \cosh S(z) \end{bmatrix} \begin{bmatrix} a(0-\varepsilon) \\ b(0-\varepsilon) \end{bmatrix} \equiv \hat{\Sigma}(S(z)) \times \begin{bmatrix} a(0-\varepsilon) \\ b(0-\varepsilon) \end{bmatrix}, \quad S(z) = \int_{0-\varepsilon}^{z} [g(z') \pm f(z')] \mathrm{d}z'. \quad (4.6)$$

If the boundary is sharp, then

$$g(z) = S_{\Delta Z}\delta(z), \quad S_{\Delta Z} = \ln \sqrt{Z_1/Z_2}, \quad f(z) = S_{\Delta n}\delta(z), \quad S_{\Delta n} = \ln \sqrt{\cos \theta_2/\cos \theta_1}, \quad (4.7)$$

so that

$$\begin{bmatrix} a(+0) \\ b(+0) \end{bmatrix} = \hat{\Sigma}(S) \begin{bmatrix} a(-0) \\ b(-0) \end{bmatrix},$$

$$S_{\mathrm{TE}} = S_{\Delta Z} + S_{\Delta n}, \qquad S_{\mathrm{TM}} = S_{\Delta Z} - S_{\Delta n}. \quad (4.8)$$

From Eqs. (2.2), (2.6), and (2.7) and (2.11) and (2.16) for each polarization the corresponding electric field component is expressed through amplitudes $a(z)$ and $b(z)$ as

$$\begin{aligned} \text{TE:} \quad & E_y(z) = -\sqrt{2Z(z)k(z)/p(z)}[a_{\mathrm{TE}}(z)e^{ik_{\mathrm{air},z}z} + b_{\mathrm{TE}}(z)e^{-ik_{\mathrm{air},z}z}], \\ \text{TM:} \quad & E_x(z) = \sqrt{2Z(z)p(z)/k(z)}[a_{\mathrm{TM}}(z)e^{ik_{\mathrm{air},z}z} + b_{\mathrm{TM}}(z)e^{-ik_{\mathrm{air},z}z}], \end{aligned} \quad (4.9)$$

and reflection coefficient $r(b \leftarrow a) \equiv r(1 \leftarrow 1)$ at $z=0$ becomes

$$r_{\mathrm{TE}} \equiv r(E_y \to E_y) = -\tanh S_{\mathrm{TE}},$$

$$r_{\mathrm{TM}} \equiv r(E_x \to E_x) = -\tanh S_{\mathrm{TM}}. \quad (4.10)$$

It is easy to check that formulas (4.10) are identical to the well-known formulas (4.1).

Thus we come to a better understanding of Fresnel reflection. It was already agreed that we characterize the lossless reflection process by the notion of real reflection strength S, such that the reflectance is $R=|r|^2=\tanh^2 S$. Our claim here is that the strength parameter for Fresnel reflection is linear sum, $S_{\mathrm{TE}}=S_{\Delta Z}+S_{\Delta n}$ for TE polarization, or linear difference, $S_{\mathrm{TE}}=S_{\Delta Z}-S_{\Delta n}$ for TM polarization, of two separate contributions. The first one $S_{\Delta Z}$ is due to the step of impedance; see Eq. (4.4). The second one $S_{\Delta n}$ is due to the step of propagation speed (of refractive index, or of propagation direction); see Eq. (4.3). The Brewster phenomenon takes place when these two contributions to the reflection strength S cancel each other precisely. In the case of the boundary between two dielectrics this Brewster cancellation takes place for TM polarization.

Consider the incidence close to normal, $\theta_1 \ll 1$, for the light coming from air to glass, $n_1=1$, $n_2=1.5$. In this case values $S_{\Delta Z}$, $S_{\Delta n}$, and $S_{\mathrm{TE,TM}}$ are rather small, and one can substitute for the hyperbolic tangent function $\tanh(x)$ its argument x. Then one can speak of adding or subtracting the contributions directly to the reflection amplitudes instead of contributions to reflection strengths S. The above formulas are illustrated below by the graphs on Figs. 2(a)–2(c).

Figure 2(a) depicts the graphs for standard Fresnel formulas for the reflection from a glass surface ($n=1.5$) back into air. Here $R_{\mathrm{TE}}(\theta)=|r_{\mathrm{TE}}(\theta)|^2$ and $R_{\mathrm{TE}}(\theta)=|r_{\mathrm{TE}}(\theta)|^2$, derived either according to the formulas (4.1), or as $\tanh^2 S$ of two reflection strengths $S_{\mathrm{TE}}=S_{\Delta Z}+S_{\Delta n}(\theta)$, $S_{\mathrm{TE}}=S_{\Delta Z}-S_{\Delta n}(\theta)$, with $S_{\Delta Z}$ and $S_{\Delta n}(\theta)$ being the contributions due, respectively, to the step of impedance and to the step of

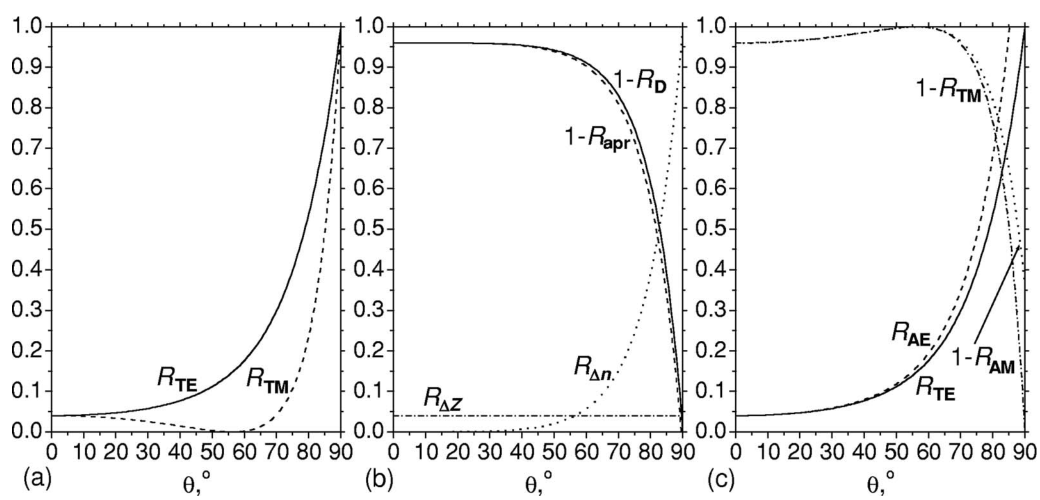

Fig. 2. Various graphs describing Fresnel reflection at the air–glass boundary versus incidence angle $\theta=\theta_{\mathrm{air}}$; see explanations in the text.

propagation speed. The two pairs of curves are absolutely identical, which supports our new physical understanding of the process of Fresnel reflection.

Figure 2(b) depicts, via the dashed–dotted horizontal straight line, the angle-independent 4% reflection coefficient $R_{\Delta Z}$ calculated as a contribution of impedance Z step only. It also depicts, via dotted curve, the assumption of equal propagation speeds in two adjacent media, and the reflection coefficient $R_{\Delta n}(\theta)$ calculated as a contribution of the propagation speed c/n step only under the assumption of equal impedances in the two media. The other two curves are presented for the transmission coefficients, $T=1-R$, to make the graph more readable. Namely, we present the exact graph for depolarized incident light, $1-R_{\mathrm{D}}(\theta)$, where $R_{\mathrm{D}}(\theta)=[R_{\mathrm{TE}}(\theta)+R_{\mathrm{TE}}(\theta)]/2$. The curve $1-R_{\mathrm{apr}}(\theta)$ deals with $R_{\mathrm{apr}}(\theta)=R_{\Delta Z}+R_{\Delta n}(\theta)$, i.e., the approximation in which we completely ignore any mutual influence of Z and n mechanisms of reflection. We see that even this approximation yields very small error: it is actually less than 3.5% in the whole angular range from 0 to 90°. That allows one to say that the description of reflection for the "depolarized world" may be done with fairly good accuracy only as a sum of reflection intensities due to the above two processes: the step of impedance (angle-independent) and the step of propagation speed (angle-dependent).

Figure 2(c) depicts the comparison of the true $R_{\mathrm{TE}}(\theta)$ and $R_{\mathrm{TM}}(\theta)$ reflection coefficients with even more approximation in their description: addition $R_{\mathrm{AM}}(\theta)$ curve and subtraction $R_{\mathrm{AE}}(\theta)$ curve, not of reflection strengths $S_{\Delta Z}+S_{\Delta n}(\theta)$ and $S_{\Delta Z}-S_{\Delta n}(\theta)$, but of reflection amplitudes $r_{\Delta Z}$ and $r_{\Delta n}(\theta)$ themselves. At larger angles the transmission $1-R_{\mathrm{AM}}(\theta)$ may become greater than one due to breakdown of this approximation. However, even this very crude approximation yields a fairly good description of these polarized components of reflection, at least in the range between 0° and 65°.

5. REFLECTION BY A VBG: EQUATIONS IN SVEA

Consider now a *nonmagnetic* (i.e., optical) medium with VBG of refractive index $n(z)$ for which

$$n(z)=n_0+n_2(z)+n_1'(z)\cos[Qz+\gamma(z)]+in_1''(z)\cos[Qz+\delta(z)],$$

$$Z(z)=\frac{Z_{\mathrm{vac}}}{n(z)}\approx\frac{Z_{\mathrm{vac}}}{n_0^2}(n_0-n_2(z)-n_1'(z)\cos[Qz+\gamma(z)]$$
$$-in_1''(z)\cos[Qz+\delta(z)]). \tag{5.1}$$

Here real $n_1'(z)$ and $\gamma(z)$ are slowly varying zero-to-top amplitude and phase of the "refractive" component of VBG, respectively; $|in_1''(z)|$ and $\delta(z)$ are the same characteristics of the "absorptive" part of VBG; and $n_2(z)$ is a small local correction to the constant real refractive index n_0. That correction $n_2(z)$ includes possible loss $i\,\mathrm{Im}[n_2(z)]$, so that the spatially averaged power attenuation coefficient of the material is $\alpha_{\mathrm{loss}}[1/\mathrm{m}]=2\omega n_2''/c$.

We can calculate our coupling functions $f(z)$ and $g(z)$ according to Eq. (2.8) by an approximate differentiation of fast oscillating terms of $n(z)$ only

$$f(z)=\frac{\frac{1}{2}n_{\mathrm{air}}^2\sin\theta_{\mathrm{air}}^2}{n^2(z)-n_{\mathrm{air}}^2\sin\theta_{\mathrm{air}}^2}\frac{\mathrm{d}\ln n(z)}{\mathrm{d}z},\quad g(z)=\frac{1}{2}\frac{\mathrm{d}\ln n(z)}{\mathrm{d}z},$$

$$\frac{\mathrm{d}\ln n(z)}{\mathrm{d}z}\approx -Q\frac{n_1'(z)\sin[Qz+\gamma(z)]+in_1''(z)\sin[Qz+\delta(z)]}{n_0}. \tag{5.2}$$

Effective interaction between waves a and b occurs at the Bragg condition, when

$$Q\approx 2\frac{\omega}{c}n_0\cos\theta_{\mathrm{in}},\quad \cos\theta_{\mathrm{in}}=\sqrt{1-n_{\mathrm{air}}^2\sin\theta_{\mathrm{air}}^2/n_0^2}. \tag{5.3}$$

Here θ_{in} is the angle between the z axis and the propagation direction of light inside the VBG. Thus Eq. (5.2) with its accounting for the Bragg condition leads to coupling functions for TE and TM polarizations

$$g(z)+f(z)\approx-\frac{\omega}{c}\frac{n_1'(z)\sin[Qz+\gamma(z)]+in_1''(z)\sin[Qz+\delta(z)]}{\cos\theta_{\mathrm{in}}},$$

$$g(z)-f(z)\approx[g(z)+f(z)]\cos 2\theta_{\mathrm{in}}. \tag{5.4}$$

We see a natural result: the coupling coefficient [which in optics is due to modulation of $\varepsilon(z)$ only] is smaller by a factor $\rho=(\mathbf{p}_a\cdot\mathbf{p}_b)=\cos 2\theta_{\mathrm{in}}$ for TM polarization in comparison with the coupling for TE polarization, where that factor equals one. Here $\mathbf{p}_a$ and $\mathbf{p}_b$ are unit polarization vectors of the electric field for waves a and b. The function $\sin[Qz+\gamma(z)]$ is written as

$$\sin[Qz+\gamma(z)]=\frac{1}{2i}[e^{iQz+i\gamma(z)}-e^{-iQz-i\gamma(z)}], \tag{5.5}$$

with a similar expression for the $\sin[Qz+\delta(z)]$. The SVE approximation, which we will use later, corresponds to keeping only one of two exponential terms from Eq. (5.5) in the coupling terms from Eqs. (2.14) and (2.17), namely, the terms that will perform z-accumulated coupling. As a result, we get the equation for matrix $\hat{M}(z)$ expressing values $a(z)$ and $b(z)$ through $a(0)$ and $b(0)$:

$$\frac{\mathrm{d}}{\mathrm{d}z}\hat{M}(z)=\hat{V}_{\mathrm{TE,TM}}(z)\hat{M}(z),$$

$$\hat{V}(z)=\begin{bmatrix} i[p(z)-k_{\mathrm{air},z}] & i\kappa_+(z)e^{iQz-2ik_{\mathrm{air},z}z} \\ -i\kappa_-(z)e^{-iQz+2ik_{\mathrm{air},z}z} & -i[p(z)-k_{\mathrm{air},z}]\end{bmatrix}, \tag{5.6}$$

$$\text{TE:}\quad \kappa_+(z)=\frac{\omega}{c}\frac{n_1'(z)e^{i\gamma(z)}+in_1''(z)e^{i\delta(z)}}{2\cos\theta_{\mathrm{in}}},$$

$$\kappa_-(z)=\frac{\omega}{c}\frac{n_1'(z)e^{-i\gamma(z)}+in_1''(z)e^{-i\delta(z)}}{2\cos\theta_{\mathrm{in}}}. \tag{5.7}$$

Interaction coefficients κ_+ and κ_- have indices (+) or (−), denoting the $\pm z$ direction in which the result of corresponding scattering propagates. These coefficients for TM polarization are smaller by the polarizational factor $\rho = \cos 2\theta_{\text{in}}$. As written in Eq. (5.6), the interaction matrix still contains fast-oscillating phase factors. However, by choosing the "central" value of the real parameter $p_0 = Q/2$, one can present the matrix $\hat{M}(z)$ in the form

$$\hat{M}(z) = \hat{K}((Q/2 - k_{\text{air},z})z)\hat{P}(z), \tag{5.8}$$

so that the equation for the $\hat{P}$ matrix becomes "slowly varying" indeed:

$$\frac{\mathrm{d}\hat{P}(z)}{\mathrm{d}z} = \hat{W}(z)\hat{P}(z),$$

$$\hat{W}(z) = \begin{bmatrix} i[p(z) - Q/2] & i\kappa_+(z) \\ -i\kappa_-(z) & -i[p(z) - Q/2] \end{bmatrix}. \tag{5.9}$$

The Bragg condition is satisfied when $p(z) = Q/2$.

As we have already discussed, the numerical (or analytic) solution of the Cauchy problem for this system placed between $z_1 = 0$ and $z_2 = L$ yields the matrix $\hat{P}(L)$, and thus $\hat{M}(L)$. If our glass plate with the VBG is placed at an arbitrary z_1 then according to the definition of amplitudes $a(z)$ and $b(z)$ the matrix $\hat{M}(z_2, z_1)$ will be

$$\hat{M}(z_2 = z_1 + L, z_1) = \hat{K}(-k_{\text{air},z}z_1)\hat{M}(L)\hat{K}(k_{\text{air},z}z_1). \tag{5.10}$$

The value of the matrix $\hat{M}(z_2, z_1)$ at $z_2 = z_1 + L$, i.e., at the end of the VBG, allows one to find reflection and transmission coefficients.

6. HOMOGENEOUS VBG: KOGELNIK'S ANALYTICAL SOLUTION

Consider a VBG medium placed between z_1 and $z_2 = z_1 + L$ with the above refractive index profile $n(z - z_1)$ with homogeneous (constant) parameters n_1', in_1'', γ, δ, n_2', in_2'', and with $\alpha_{\text{loss}}[1/\text{m}] = 2\omega n_2''/c$ being the spatially averaged attenuation coefficient for power. For definiteness we consider here the TE polarization. Function $p(z)$ from Eq. (2.8) in this case will be assumed constant with small positive imaginary part:

$$p = \frac{\omega}{c}\sqrt{(n_0 + n_2)^2 - n_{\text{air}}^2 \sin \theta_{\text{air}}^2} \approx \frac{\omega}{c}\left(n_0 \cos \theta_{\text{in}} + \frac{n_2}{\cos \theta_{\text{in}}}\right)$$

$$= p' + ip'', \quad p'' = \frac{\alpha_{\text{loss}}}{2 \cos \theta_{\text{in}}}. \tag{6.1}$$

After that, the $\hat{W}$ matrix from Eqs. (5.9) becomes z-independent, and Eqs. (5.9) get the explicit solution described in Appendix B:

$$\frac{\mathrm{d}\hat{P}(z)}{\mathrm{d}z} = \hat{W}\hat{P}(z), \quad \hat{W} = \begin{bmatrix} i\Delta & i\kappa_+ \\ -i\kappa_- & -i\Delta \end{bmatrix} \Rightarrow$$

$$\hat{P}(L) = e^{\hat{W}L} = \hat{1}\cosh G + \hat{W}L\frac{\sinh G}{G}, \tag{6.2}$$

$$\Delta = p' + ip'' - Q/2, \quad G = \sqrt{S_+S_- - X^2},$$

$$S_\pm = \kappa_\pm L = \kappa_\pm' L + i\kappa_\pm'' L, \quad X = \Delta L = X' + iX''. \tag{6.3}$$

The dimension of Δ is [1/m]. In this manner, the matrix $\hat{P}(L)$ becomes

$$\hat{P}(L) = \begin{bmatrix} \cosh G + iX\dfrac{\sinh G}{G} & iS_+\dfrac{\sinh G}{G} \\ -iS_-\dfrac{\sinh G}{G} & \cosh G - iX\dfrac{\sinh G}{G} \end{bmatrix}. \tag{6.4}$$

Going back to Eqs. (5.10) and (5.8), we obtain the expression for the matrix $\hat{M}(z_2, z_1)$:

$$\hat{M}_{\text{VBG}}(z_2, z_1) = \hat{K}\left(-k_{\text{air},z}z_1 - \left(k_{\text{air},z} - \frac{Q}{2}\right)L\right)\hat{P}(L)\hat{K}(k_{\text{air},z}z_1). \tag{6.5}$$

As a result, the reflection coefficient by a VBG becomes

$$r(b \leftarrow a) = r_- = -\frac{M_{ba}}{M_{bb}} = -e^{2ik_{\text{air},z}z_1}\frac{P_{ba}}{P_{bb}}$$

$$= e^{2ik_{\text{air},z}z_1}\frac{iS_- \sinh G/G}{\cosh G - iX \sinh G/G}. \tag{6.6}$$

Formulas (6.1)–(6.6) allow one to find the reflection coefficient even in the presence of loss or gain. Equivalent results in different notations were first derived in the fundamental work by H. Kogelnik [6]. We have rederived them in our notations, which facilitate the subsequent account of Fresnel reflections.

Imaginary detuning may be expressed via intensity attenuation coefficient α_{loss} (3.16):

$$X'' \equiv Y = \frac{\alpha_{\text{loss}}L}{2 \cos \theta_{\text{in}}}. \tag{6.7}$$

The relatively difficult part is to express the dimensionless quantity $\mathrm{Re}(X)$ via observables; this parameter signifies the detuning from the Bragg condition (Bc). Suppose that Bc is satisfied exactly at certain values of incident angle $\theta_{\text{air},0}$ at the wavelength $\lambda_{\text{vac},0}$ for definite values of Q and n_2. Then in the case of relatively small (but homogeneous) deviations from Bc one gets

$$\delta\,\mathrm{Re}(X) = \frac{-2\pi n_0 L \cos \theta_{\text{in}}}{\lambda_{\text{vac},0}}\left\{\frac{\delta\lambda_{\text{vac}}}{\lambda_{\text{vac}}} + \frac{\delta Q}{2Q} + \frac{1}{\cos^2 \theta_{\text{in}}}\left[\frac{n_{\text{air}}^2}{2n_0^2}\left(\sin^2 \theta_{\text{air}} - \sin^2 \theta_{\text{air},0}\right) - \frac{\delta n_2}{n_0}\right]\right\}. \tag{6.8}$$

While the look of the expressions (6.1)–(6.8) is rather heavy, their calculation by any computer is quite straightforward. Moreover, the accuracy of modern computers allows one to use a procedure that is morally reprehensible, but numerically admissible: Calculate $p(\text{detuned}) - p(\text{Bc})$ as the small difference of two large quantities. Such procedure reduces the risk of making a typographical error in Eq. (6.8).

In the absence of loss or gain and with modulation of real $\mathrm{Re}(n_1')$ one gets $S_+ = (S_-)^* = S_0 e^{i\gamma}$, $\mathrm{Im}(X)=0$, so one can use the notion of reflection strength S, and then the reflection coefficient $R_{\mathrm{VBG}} = R_+ = R_- = R$ becomes

$$R = |r(b \leftarrow a)|^2 = \tanh^2 S = \frac{\sinh^2 G}{\cosh^2 G - X^2/S_0^2};$$

$$S = \operatorname{arcsinh}\left(S_0 \frac{\sinh G}{G}\right), \quad S_0 = \sqrt{S_+ S_-} = |S_+|,$$

$$G = \sqrt{S_0^2 - X^2}. \tag{6.9}$$

Finally, at exact Bc, $X=0$, and without loss, the reflection strength S is

$$S = S_0 = \frac{\pi n_1 L}{\lambda_{\mathrm{vac}} \cos \theta_{\mathrm{in}}}, \tag{6.10}$$

which constitutes the most important and simplest formula of Kogelnik's VBG theory.

7. INFLUENCE OF FRESNEL REFLECTIONS

For a sharp boundary positioned at z_{b}, the process of Fresnel reflection of the waves with TE and TM polarizations is described, according to Eqs. (4.1)–(4.9) and (5.10), by the matrix $\hat{M}$:

$$\hat{M}(z_{\mathrm{b}}+0, z_{\mathrm{b}}-0) = \hat{K}(-k_{\mathrm{air},z} z_{\mathrm{b}}) \hat{\Sigma}(S) \hat{K}(k_{\mathrm{air},z} z_{\mathrm{b}}), \tag{7.1}$$

$$\hat{\Sigma}(S) = \begin{bmatrix} \cosh S & \sinh S \\ \sinh S & \cosh S \end{bmatrix},$$

$$S_{\mathrm{TE,TM}} = \ln \sqrt{\frac{Z_1}{Z_2}} \pm \ln \sqrt{\frac{\cos \theta_2}{\cos \theta_1}} \equiv \ln \sqrt{\frac{n_2}{n_1}} \pm \ln \sqrt{\frac{\cos \theta_2}{\cos \theta_1}}. \tag{7.2}$$

At incidence normal to the boundary between two optical media with n_1 and n_2, the reflection strength is the same for both polarizations: $S = 1/2 \ln(n_2/n_1)$, since $Z = Z_{\mathrm{vac}}/n$. For the particular case $n_2/n_1 = 1.5$ one gets $S = 0.2027$.

Now consider the case of a VBG positioned between z_1 and $z_2 = z_1 + L$ with background refractive index n_0; this VBG is surrounded by air, and $n_{\mathrm{air}} = 1$. For a VBG with boundaries, the transformation matrix $\hat{M}$ given by Eq. (6.5) will be surrounded by two boundary matrices of the type of Eq. (7.1):

$$\hat{M}(z_2+0, z_1-0) = \hat{K}(-k_{\mathrm{air},z} z_2) \hat{\Sigma}(S_2) \hat{K}(k_{\mathrm{air},z} z_2) \hat{M}_{\mathrm{VBG}}(z_2, z_1) \times \hat{K}(-k_{\mathrm{air},z} z_1) \hat{\Sigma}(S_1) \hat{K}(k_{\mathrm{air},z} z_1). \tag{7.3}$$

Here S_1 and S_2 are the strengths of reflections at the corresponding boundaries, and the matrix $\hat{M}_{\mathrm{VBG}}(z_2, z_1)$ is given by Eq. (6.5). While analytical expressions look quite heavy, one has to multiply the matrices given by explicit expressions only; such a procedure is very simple for a computer.

In the case of a perfectly lossless VBG one has to take into account the phase relationships between contributions of the first boundary, the VBG, and the second boundary. After summation of arguments in corresponding $\hat{K}$ matrices the total matrix of VBG with boundaries (7.3) will be

$$\hat{M} = \hat{K}(-k_{\mathrm{air},z} z_2) \hat{\Sigma}(S_2) \hat{K}((\gamma + QL)/2) \times \hat{P}_{S_0,X} \hat{K}(-\gamma/2) \hat{\Sigma}(S_1) \hat{K}(k_{\mathrm{air},z} z_1), \tag{7.4}$$

$$\hat{P}_{S_0,X} = \begin{bmatrix} \cosh G + iX \dfrac{\sinh G}{G} & iS_0 \dfrac{\sinh G}{G} \\ -iS_0 \dfrac{\sinh G}{G} & \cosh G - iX \dfrac{\sinh G}{G} \end{bmatrix},$$

$$G = \sqrt{S_0^2 - X^2}, \tag{7.5}$$

with S_0 and X defined in Eqs. (6.3) and (6.9). We see that the character of the curve of reflectance versus detuning depends on two phases, γ and QL, both related to the properties of the specimen that contains the grating. Their values fluctuate from one specimen to another as a result of manufacturing of the VBG. Quite often the specimens are coated with antireflection layers.

Far from the resonance when $X \gg S_0$ the matrix $\hat{P}_{S_0,X}$ will transform into diagonal phase matrix $\hat{K}(X)$. Then after summation of phases between boundaries we simplify the matrix (7.4) to

$$\hat{M} = \hat{K}(-k_{\mathrm{air},z} z_2) \hat{\Sigma}(S_2) \hat{K}(\varphi) \hat{\Sigma}(S_1) \hat{K}(k_{\mathrm{air},z} z_1),$$

$$\varphi = pL = \frac{\omega}{c} n_0 L \cos \theta_{\mathrm{in}}, \tag{7.6}$$

which describes ordinary glass plate with interferometric properties defined by phase difference pL. When this relative boundary phase is equal to a (typically large) integer number m of π, then matrix $\hat{K}(\varphi)$ is proportional to a unit matrix, and the total reflection strength is $S = S_1 + S_2 = 0$. This corresponds to perfect resonant transmission of a Fabry–Perot interferometer based on reflections by two boundaries. If at some particular frequency/angle point our VBG has zero strength, e.g., if $G = im\pi$, with m being an integer nonzero number, then $\hat{P}_{S_0,X}$ is proportional to a unit matrix and again boundary strength matrices $\hat{\Sigma}$ are separated by a phase matrix so total reflectance will be defined only by boundaries.

Let us go back to the VBG without background loss or gain, and with boundaries of different reflectances R_1 and R_2 in the general case, so their reflection strengths are $|S_{1,2}| = \operatorname{arctanh} \sqrt{R_{1,2}}$, respectively. Multiplication of the corresponding matrices of the first boundary, of the VBG, and of the second boundary yielded the resulting matrix (7.4). Maximum and minimum values of the total resultant strength are realized when boundary terms are added or subtracted from the VBG term; that is,

$$R = \tanh^2 S, \quad S_{\max} = S_{\mathrm{VBG}} + |S_1| + |S_2|,$$

$$S_{\min} = S_{\mathrm{VBG}} - (|S_1| + |S_2|), \qquad (7.7)$$

due to appropriate intermediate phases. *We consider formulas (7.7) one of the important results of the present work.*

Figure 3 was obtained by honest multiplication of relevant matrices, and then by depicting all possible values of $|R_{\mathrm{total}}|^2$ at various combinations of phases. We see that in the region of perfect Bc, $X=0$, reflectivity is not affected strongly by the boundaries. Even if one has to deal with Fresnel reflections, $R_1=R_2=0.04$ (for $n_0=1.5$), the modified reflection at exact Bc is within the limits $0.9779 \le R_{\mathrm{total}} \le 0.9956$ for $R_{\mathrm{VBG}}=0.99$ ($S=2.993$). On the other hand, in the spectral points of exactly zero R_{VBG}—where, in Eq. (6.9), $X^2=S_0^2+m^2\pi^2$ with integer nonzero m—the residual reflection varies within the interval

$$\tanh^2(S_1 - S_2) \le R \le \tanh^2(S_1 + S_2), \quad S_i = \operatorname{arctanh}\sqrt{R_i}. \qquad (7.8)$$

In particular, if $R_1=R_2=0.04$ then $0 \le R \le 0.1479$. Another example is $R_1=R_2=0.003$; then $0 \le R \le 0.0119$.

8. EXPERIMENTAL DEMONSTRATION OF PHASE-SHIFTED VBG AND FABRY–PEROT EFFECTS

Consider a VBG made of two equally strong parts, each of them having the same values S_{VBG}. Then the reflection action of the compound VBG depends on the mutual phases of these two gratings. If there is no phase shift $\Delta\gamma$ between cosinusoidal modulations of refractive index inside these two gratings, then the combined VBG merely acquires double strength $S_{\mathrm{tot}}=2S_{\mathrm{VBG}}$. However, any intermediate shift, $0<\Delta\gamma<2\pi$, yields a narrow spectral transmission peak (or reflection dip) to $T=1$ ($R=0$). The physical sense of this 100% transmission peak is similar to the

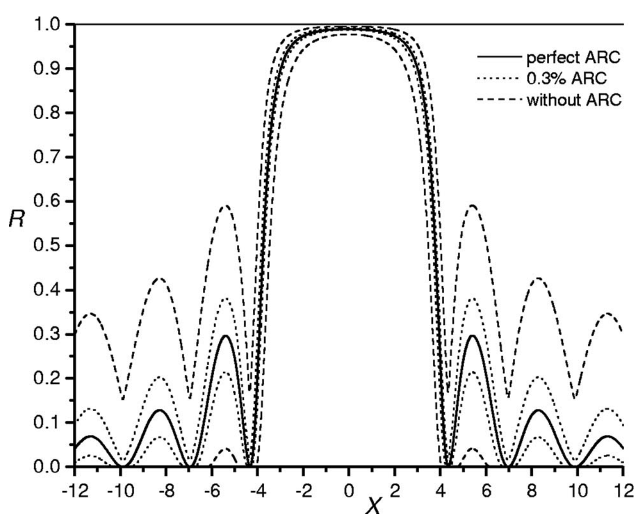

Fig. 3. Reflectivity R of a VBG with account of interference of reflection by the VBG proper with two extra contributions: from the two boundaries of the specimen, for all possible phase combinations. Values of R are between the dashed curves for Fresnel 4% reflections from bare boundaries, and are between the dotted curves for antireflection coatings (ARC) at 0.3% each.

100% transmission peak of a Fabry–Perot resonator with flat mirrors, when the resonant condition is satisfied.

In order to describe such a configuration of two VBGs, we consequently have to multiply matrices of elements with corresponding phases. We actually performed the experimental study of two uncoated identical reflective VBGs placed very close to each other with small gap l between them filled by immersion liquid with the same background refractive index n_0. The coordinates of the first grating boundaries were $z_0=0$ and $z_1=L$, and the second grating was positioned between $z_2=L+l$ and $z_3=2L+l$. Spectral parameters X and strengths S_0 were the same for both gratings, but initial phases γ_1 and γ_2 were different. Boundary reflection strength from air to glass was S_{b} and that from glass to air was $-S_{\mathrm{b}}$. The transformation matrix determining waves a and b after this compound system, $z>z_3$, through values of a and $b$before it, $z<0$, is a product of matrices of two types—Eqs. (7.1) and (6.5)—with $\hat{P}(L)=\hat{K}(\gamma/2)\hat{P}_{S_0,X}\hat{K}(-\gamma/2)$; see also Eq. (7.5). After simplification of phase arguments it becomes

$$\begin{bmatrix} a(z_3+0) \\ b(z_3+0) \end{bmatrix} = \hat{M} \begin{bmatrix} a(-0) \\ b(-0) \end{bmatrix},$$

$$\hat{M} = \hat{K}(\beta_3)\hat{\Sigma}(-S_{\mathrm{b}})\hat{K}(\beta_2)\hat{P}_{S_0,X}\hat{K}\left(\frac{\Delta\gamma}{2}\right)\hat{P}_{S_0,X}\hat{K}(\beta_1)\hat{\Sigma}(S_{\mathrm{b}}),$$

$$\beta_1 = -\frac{\gamma_1}{2}, \quad \beta_2 = \frac{\gamma_2 + QL}{2}, \quad \beta_3 = -k_{\mathrm{air},z}z_3,$$

$$\Delta\gamma = QL + \gamma_1 - \gamma_2 + 2pl. \qquad (8.1)$$

For small size l of the gap phase, pl (or kl at normal incidence) is approximately the same for all wavelengths in question. We see that the reflection characteristics of this compound system depend on three intermediate phases: phase shift $\Delta\gamma$ between two cosinusoidal modulations in VBGs contacted via immersion layer and two outside boundary phases β_1 and β_2.

We present the experimental demonstration of the coherent combination of two π-shifted VBGs in air. The VBGs used for this demonstration were recorded inside PTR glass [9,10]. They have central wavelength at 1063.4 nm, thickness of 2.76 mm, and refractive index modulation of 154 ppm (middle-to-top). They were recorded inside PTR glass without slant, and diffraction efficiency was equal to 72%, so $S_0=1.25$. The two VBGs were fixed on mirror holders, and one holder was motorized with a piezoelectric transducer that allowed fine translation and fine angle tuning. The setup for the measurement of the spectral response used a tunable laser having a 1 pm resolution; see Fig. 4.

The laser radiation was spatially filtered by a single-mode fiber and coupled to a collimator. The 1 mm diameter output beam was sent through the VBG assembly, and the transmitted signal was measured using a silicon-amplified photodiode associated with a data acquisition card. To adjust the parallelism of the two VBGs, a fiber coupler was used between the laser and the collimator. Another silicon-amplified photodiode was used to mea-

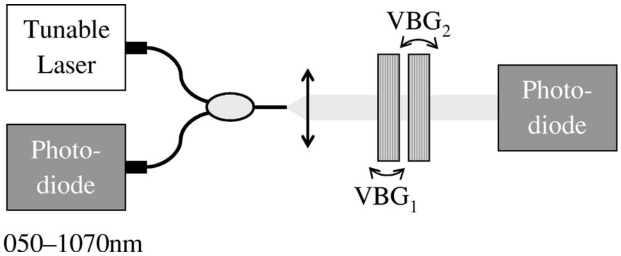

Fig. 4. Experimental setup for the coherent combination of two VBGs in PTR glass.

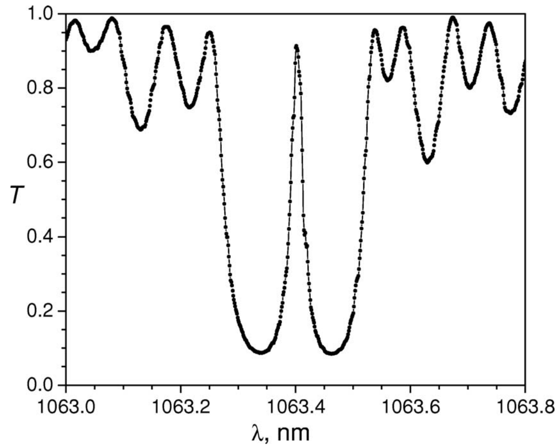

Fig. 5. Experimental transmission of two π-shifted VBGs.

sure the power reflected from the combined VBGs and recoupled inside the collimator.

It is important to stress that the laser used for the measurement was combined with a circulator that blocked all

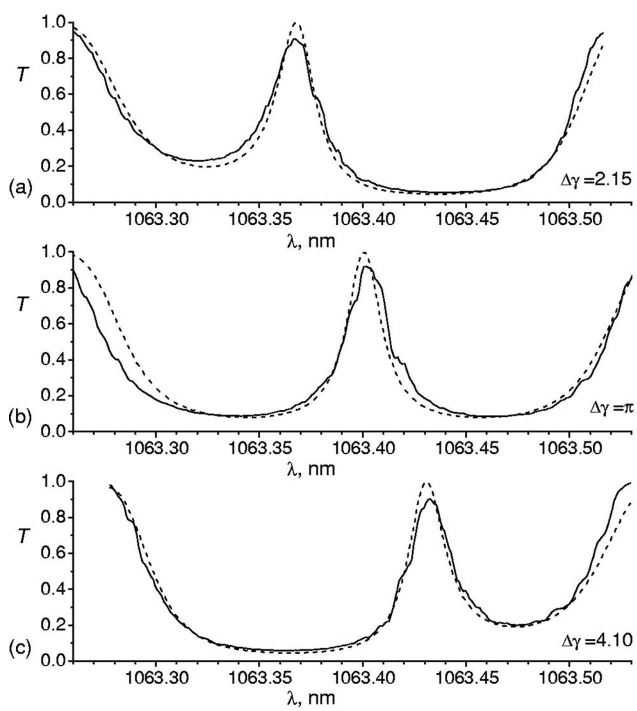

Fig. 6. Spectral shift of resonant transmission due to phase shift $\Delta\gamma$ between two grating modulations; see the text for details.

reflected signal that otherwise would have been reinjected inside the laser cavity and would lock it to the wavelength of the filter. Using this coupler, the two VBGs could thus be aligned by autocollimation.

Typical spectral dependence of the transmission of the filter is shown in Fig. 5. Oscillations in transmission outside the resonance are due to the phase interplay between uncovered Fresnel reflections and secondary evanescent lobes of the gratings. This filter presents a transmission higher than 90%. Bandwidth was $\approx$25 pm (FWHM) and rejection width was 200 pm. Rejection outside the resonance was better than 10 dB and could be improved by combining it with an additional VBG or using VBGs with higher diffraction efficiencies [18].

To illustrate the principle of phase matching between the two VBG, we changed the distance between them and recorded the transmission for each distance; see Figs. 6(a)–6(c). One can see that according to the distance between the two VBG, the resonance moved inside the main lobe of the diffraction efficiency of the VBG. When the distance was optimized and phase shift $\Delta\gamma$ was equal to π, the resonance was centered in the middle of this lobe. When this phase was different from π, resonance was shifted to the edge of the lobe.

The solid curves at Figs. 6(a)–6(c) correspond to experimental data, while the dashed curves are theoretical fits with optimized $\Delta\gamma$ for these actual gratings. We see reasonable agreement of theory with experiment.

9. CONCLUSION

In this paper we reworked the matrix approach to the calculation of reflection and transmission of electromagnetic waves by a layered medium. Two principal results of our theory should be mentioned. First, we were able to elucidate separate contributions of the gradient of impedance $Z=(\mu/\varepsilon)^{1/2}$, and of the gradient of propagation speed $v=1/(\mu\varepsilon)^{1/2}$. Second, for lossless media we introduced the notion of reflection strength S such that the reflection coefficient $R=(\tanh S)^2$. We established the law of composition of sequential reflection elements, which depends on the phase difference between their contributions. At zero phase difference one has simply to add individual strengths, while at phase difference π one subtracts these strengths.

Our findings constitute a new understanding of long-established Fresnel formulas for reflection by a sharp boundary between two media. Namely, the strengths $S_{\mathrm{TE}}(\theta)$ and $S_{\mathrm{TM}}(\theta)$ are shown to be a sum, $S_{\mathrm{TE}}(\theta)=G+F(\theta)$, and a difference, $S_{\mathrm{TM}}(\theta)=G-F(\theta)$, of two contributions. One of them is an angle-independent contribution of the impedance step, while the other is an angle-dependent contribution of the step of propagation speed.

We demonstrated experimentally a Fabry–Perot-type spectral filter with bandwidth of transmission peak $\Delta\lambda(\mathrm{FWHM})\approx 25$ pm in the range $\lambda_{\max}-\lambda_{\min}\approx 200$ pm.

APPENDIX A: PECULIAR PROPERTIES OF VBG MADE WITH USE OF ABSORBING MATERIALS

Consider the diffraction of light by a VBG made of nonmagnetic medium, where $\mu=\mu_{\mathrm{vac}}$, and where both the grating and the background dielectric permittivity contain real (refractive) and imaginary (absorptive) parts. Actually, even in the absence of physical absorption in the PTR glass, scattering by microcrystals may produce effective $\mathrm{Im}[\delta n(z)]$.

The first mathematical statement to be made is that it is impossible to have a spatial Fourier component of the absorptive grating without background absorption. Consider a Fourier representation of a periodic profile of the imaginary part of the refractive index assuming $|\mathrm{Im}(\delta n)| \ll n_0$:

$$n''(z) = \langle n'' \rangle + \sum_{l=+1}^{+\infty} (c_l e^{ilQz} + c_l^* e^{-ilQz}). \tag{A1}$$

Then the requirement of $n''(z)$ being nonnegative everywhere (absence of gain) limits the modulus $|c_l|$ of the amplitude of the Fourier component with $l=1$:

$$|c_1| \le \langle n'' \rangle. \tag{A2}$$

The inequality becomes equality when the spatial profile of $n''(z)$ is a periodic series of infinitely thin positive delta functions:

$$n''(z) = \langle n'' \rangle D \sum_{m=-\infty}^{+\infty} \delta(z - mD) = \sum_{l=-\infty}^{+\infty} \langle n'' \rangle e^{ilQz}, \quad D = \frac{2\pi}{Q}. \tag{A3}$$

For combined complex refractive index with real cosinusoidal modulation and thin-layer-type dependence of the imaginary part,

$$n(z) = n_0 + n_1' \cos(Qz + \gamma) + in''(z), \tag{A4}$$

the coupling coefficients analogous to Eq. (5.7) for counterpropagating waves a and b will be

$$\kappa_{\pm,\mathrm{TE}} = \frac{\omega}{c \cos \theta_{\mathrm{in}}} \left(\frac{1}{2} n_1' e^{\pm i\gamma} + i \langle n'' \rangle \right),$$

$$\kappa_{\pm,\mathrm{TM}} = \kappa_{\pm,\mathrm{TE}} \rho, \quad \rho = \cos(2\theta_{\mathrm{in}}). \tag{A5}$$

The most surprising phenomenon takes place for the VBG made out of thin, periodic, purely absorbing layers only. Then Eqs. (6.2) at the Bc with residual imaginary detuning $\Delta = ip''$ for TE polarization become

$$\frac{\mathrm{d}\hat{P}(z)}{\mathrm{d}z} = \hat{W}\hat{P}(z), \quad \hat{W} = \begin{bmatrix} i\Delta & i\kappa_+ \\ -i\kappa_- & -i\Delta \end{bmatrix}$$

$$= p'' \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \quad p'' = \frac{\langle n'' \rangle \omega}{c \cos \theta_{\mathrm{in}}} = \frac{\alpha_{\mathrm{loss}}}{2 \cos \theta_{\mathrm{in}}},$$

$$\hat{P}(L) = \exp(\hat{W}L) = \hat{1} + \hat{W}L = \begin{bmatrix} 1 - Y & -Y \\ Y & 1 + Y \end{bmatrix},$$

$$Y = p''L = \frac{\alpha_{\mathrm{loss}} L}{2 \cos \theta_{\mathrm{in}}}, \tag{A6}$$

see Appendix B. The resultant reflection coefficients $R_+ = R(a \leftarrow b)$ and $R_- = R(b \leftarrow a)$ are the same and are equal to

$$R_+ = R_- = R = \left(\frac{Y}{1 + Y} \right)^2, \quad T = \frac{1}{(1 + Y)^2},$$

$$1 - R - T = \frac{2Y}{(1 + Y)^2}. \tag{A7}$$

In a curious way, the reflection increases asymptotically to 100% as $Y \to \infty$, i.e., when the total thickness of such a purely absorptive VBG goes to infinity (albeit it increases rather slowly). This phenomenon is similar to the Bormann effect of anomalously high transmission of X rays in Bc in crystals for TE polarization. This phenomenon was suggested as a candidate for making artificial X-ray mirrors in [19].

Another interesting example is when such a concentrated modulation of absorption is accompanied by the modulation of refraction. One can adjust the phase γ and amplitude of $\mathrm{Re}(n_1)$ in such a manner that $n_1' e^{i\gamma} = 2i \langle n'' \rangle$ and the interaction matrix $\hat{W}$ becomes

$$\hat{W} = \frac{\alpha_{\mathrm{loss}}}{2 \cos \theta_{\mathrm{in}}} \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}. \tag{A8}$$

In other words, 90°-phase-shifted gratings of $\mathrm{Re}(n)$ and of $\mathrm{Im}(n)$ enhance each other for the $b \to a$ scattering, but completely compensate each other's influence for $a \to b$. In that case for TE polarization

$$\exp(\hat{W}L) = \hat{1} \cosh Y + \hat{W}L \frac{\sinh Y}{Y} = \begin{bmatrix} e^{-Y} & -2 \sinh Y \\ 0 & e^{Y} \end{bmatrix},$$

$$Y = \frac{\alpha_{\mathrm{loss}} L}{2 \cos \theta_{\mathrm{in}}}, \tag{A9}$$

$$R_+ = 0, \quad R_- = (1 - e^{-2Y})^2, \quad T = e^{-2Y},$$

$$1 - T - R_+ = 1 - e^{-2Y}, \quad 1 - T - R_- = e^{-2Y}(1 - e^{-2Y}). \tag{A10}$$

In this case maximum $R(b \leftarrow a)$ reflection may reach 100%. Maximum absorption under illumination by the b wave is 100%; it is reached at $Y \gg 1$. Maximum absorption under illumination by the a wave is 25%; it is reached at $Y = 0.5 \ln 2 = 0.347$. However, in a remarkable way, the reflection $R(a \leftarrow b) = R_+$ is identically zero: quite a surprise!

APPENDIX B: LAGRANGE FORMULA FOR A FUNCTION OF A MATRIX

Here are the heuristic considerations justifying the so-called "Lagrange interpolation formula" for the function of a matrix. It should be emphasized that it is an *exact* formula, in spite of traditional use of the adjective "interpolation." One can find detailed proofs in almost any textbook on matrices; see, e.g., [20].

Consider a function $F(z)$ of one (generally complex) variable z. Suppose one wants to interpolate this function by a polynomial of nth power:

$$F(z) \approx P_n(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0. \tag{B1}$$

To do this, one can use $n+1$ values of the function $F(z_j)$ at different points of the argument z_j, where $j=1,2,\ldots,n+1$. The most compact expression for such polynomial $P_n(z)$, which is based on the above data, constitutes the "Lagrange interpolation formula." For brevity we write it for the cases $n=1$ and $n=2$ only:

$$F(z) \approx P_1(z) = F(z_1)\frac{z-z_2}{z_1-z_2} + F(z_2)\frac{z-z_1}{z_2-z_1},$$

$$F(z) \approx P_2(z) = F(z_1)\frac{(z-z_2)(z-z_3)}{(z_1-z_2)(z_1-z_3)} + F(z_2)\frac{(z-z_1)(z-z_3)}{(z_2-z_1)(z_2-z_3)} + F(z_3)\frac{(z-z_1)(z-z_2)}{(z_3-z_1)(z_3-z_2)}. \tag{B2}$$

Indeed, the design of the "fractions" in relations (B2) is such that their values at appropriate points z_j are either 0 or 1.

Going to the functions of matrices, one can use the remarkable Cayley–Hamilton theorem, which states that any power of n-by-n matrix $\hat{Z}$ may be expressed as a linear combination of unit matrix $\hat{1} \equiv (\hat{Z})^0$, matrix $(\hat{Z})^1,\ldots,$ and up to $(\hat{Z})^{n-1}$ inclusive. That is the motivation to assume the formula of the type (B1) with $(n-1)$-st power of the polynomial *exact*.

Following is the explicit expression for the function of a 2-by-2 matrix $\hat{Z}$; let λ_1 and λ_2 be the eigenvalues of our matrix $\hat{Z}$, i.e., the roots of the characteristic equation

$$\det(\hat{Z} - \lambda\hat{1}) = \lambda^2 - (Z_{11}+Z_{22})\lambda + Z_{11}Z_{22} - Z_{12}Z_{21} = 0; \Rightarrow$$

$$\Rightarrow \quad \lambda_{1,2} = \frac{1}{2}(Z_{11}+Z_{22}) \pm \sqrt{\frac{1}{4}(Z_{11}-Z_{22})^2 + Z_{12}Z_{21}}. \tag{B3}$$

Then

$$F(\hat{Z}) = F(\lambda_1)\frac{\hat{Z}-\lambda_2\hat{1}}{\lambda_1-\lambda_2} + F(\lambda_2)\frac{\hat{Z}-\lambda_1\hat{1}}{\lambda_2-\lambda_1} \equiv \left[\frac{F(\lambda_2)\lambda_1 - F(\lambda_1)\lambda_2}{\lambda_1-\lambda_2}\right]\hat{1} + \left[\frac{F(\lambda_1)-F(\lambda_2)}{\lambda_1-\lambda_2}\right]\hat{Z}. \tag{B4}$$

We are most interested in the exponential function of a 2-by-2 zero-trace-matrix $\hat{Z}$. In that case

$$Z_{22} = -Z_{11}, \quad \lambda_1 = -\lambda_2 = \lambda = \lambda' + i\lambda'' = \sqrt{(Z_{11})^2 + Z_{12}Z_{21}}, \tag{B5}$$

$$\exp(\hat{Z}t) = \hat{1}\cosh(\lambda t) + \hat{Z}\frac{\sinh(\lambda t)}{\lambda}. \tag{B6}$$

Three points are worth special mention. First, both $\cosh(\lambda t)$ and $\sinh(\lambda t)/\lambda$ are even functions of λ, and therefore the particular choice of the branch of the root in Eqs. (B5) and (B6) is not important; it only must be the same in both numerator and denominator of $\sinh(\lambda t)/\lambda$. Second, it is nice that one does not have to calculate eigenvectors of the matrix $\hat{Z}$. Third, if the eigenvalue λ is close to zero or just zero, one has to use l'Hôspital's rule:

$$\exp(\hat{Z}t) \to \hat{1} + \hat{Z}t, \quad \text{if } \lambda \to 0, \quad \text{arbitrary } t. \tag{B7}$$

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