Angular and radial mode analyzer for optical beams

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We describe an approach to determining *both* the angular *and* the radial modal content of a scalar optical beam in terms of optical angular momentum modes. A modified Mach–Zehnder interferometer that incorporates a spatial rotator to determine the angular modes and an optical realization of the fractional Hankel transform (fHT) to determine the radial modes is analyzed. Varying the rotation angle and the order of the fHT produces a two-dimensional (2D) interferogram from which we extract the modal coefficients by simple 2D Fourier analysis.

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The recognition that certain electromagnetic beams have well-defined orbital angular momentum (OAM) [1] has led to intense research over the past two decades [2,3]. OAM beams are invariant upon propagation, and are, hence, useful as optical information carriers. An optical beam having OAM of $\ell\hbar$ per photon takes the form of a helical angular phase and a Laguerre–Gaussian (LG) function in the radial direction. Only the angular modes have been considered extensively heretofore, while the radial modes have been overlooked, in part due to lack of approaches to analyzing a beam into radial modes, while several OAM-mode-analysis schemes exist [4–6]. The ability to manipulate the radial modes in addition to the angular modes allows for a larger state space for encoding information in the beam.

In this Letter we describe an interferometric scheme that analyzes a scalar optical beam simultaneously in terms of angular *and* radial modes and determines the percentage of the total power contributed by these modes. The scheme makes use of the fractional Hankel transform (fHT) whose eigenfunctions are LG functions. This scheme allows us to distinguish among several classes of multimode beams, such as beams having correlated, anticorrelated, or separable OAM and radial modes.

The field distribution at the waist ($z = 0$) of a beam having OAM of $\ell\hbar$ per photon and LG-mode radial distribution is

$$u_{\ell p}(\rho,\theta) = A_{\ell p}\rho^{|\ell|}L_p^{|\ell|}(\rho^2)e^{-\rho^2/2}e^{i\ell\theta}, \tag{1}$$

where θ is the angular coordinate, $\rho = \sqrt{2}r/w$ is a normalized radial coordinate, r is the radial coordinate, w is the beam waist, p is the radial index, $L_p^{|\ell|}$ is the associated Laguerre polynomial, and $A_{\ell p} = \sqrt{p!/(\pi(p+|\ell|)!)}$ is a normalization constant such that $\iint |u_{\ell p}(\rho,\theta)|^2\rho\mathrm{d}\rho\mathrm{d}\theta = 1$. The set $\{u_{\ell p}(\rho,\theta)\}$ is complete over L^2, $\sum_{\ell,p}u_{\ell p}(\rho,\theta)u^*_{\ell p}(\rho',\theta') = \delta(\rho-\rho')\delta(\theta-\theta')$, and orthonormal, $\int u_{\ell p}(\rho,\theta)u^*_{\ell' p'}(\rho,\theta)\rho\mathrm{d}\rho\mathrm{d}\theta = \delta_{p,p'}\delta_{\ell,\ell'}$. This allows for a unique decomposition of a scalar beam:

$$E(\rho,\theta) = \sum_{\ell,p}c_{\ell p}u_{\ell p}(\rho,\theta), \tag{2}$$

where $c_{\ell p} = \iint u^*_{p\ell}(\rho,\theta)E(\rho,\theta)\rho\mathrm{d}\rho\mathrm{d}\theta$. Normalizing the total power $\iint |E(\rho,\theta)|^2\rho\mathrm{d}\rho\mathrm{d}\theta = 1$ results in $\sum_{\ell p}|c_{\ell p}|^2 = 1$.

We start by examining beams restricted to superpositions of OAM states with the radial index p set to 0, $E(\rho,\theta) = \sum_\ell c_{\ell 0}u_{\ell 0}(\rho,\theta)$, where $u_{\ell 0}(\rho,\theta) = A_{\ell 0}\rho^{|\ell|}e^{-\rho^2/2}e^{i\ell\theta}$ and $A_{\ell 0} = 1/\sqrt{\pi|\ell|!}$. We may find the weights $|c_{\ell 0}|^2$, and, hence, the angular modal spectrum, by using the arrangement in Fig. 1(a), which consists of a balanced Mach–Zehnder interferometer (MZI) including a spatial rotator in one arm. The MZI normalized output power is

$$\begin{aligned} P(\varphi) &= 1 + \Re\int_0^\infty\int_0^{2\pi}E(\rho,\theta)E^*(\rho,\theta-\varphi)\rho\mathrm{d}\rho\mathrm{d}\theta \\ &= 1 + \sum_\ell |c_{\ell 0}|^2\cos(\ell\varphi), \end{aligned} \tag{3}$$

where $0 \leq \varphi < 2\pi$ is the rotation angle, and $\Re$ refers to taking the real part. Note that the angular spectrum is discrete since φ varies periodically (period 2π). The detector integrates over the area of the beam and spatially resolved measurements are, thus, not required [7]. The Fourier transform (FT) of the interferogram $P(\varphi)$ with respect to φ yields the weights $|c_{\ell 0}|^2$. The constant term may also be eliminated by taking the difference between the signals detected at the two output ports of the MZI.

A corresponding scheme for analyzing the beam into its radial modes has not been forthcoming. There is an important feature of the above-described scheme for OAM analysis that will guide us to constructing a *radial mode analyzer*: $\{e^{i\ell\theta}\}$ are eigenmodes of spatial rotation with eigenvalues $e^{-i\ell\varphi}$, where φ is the rotation angle. The completeness of the set $\{e^{i\ell\theta}\}$ allows one to write the spatial rotation transformation $R(\theta,\theta';\varphi)$,

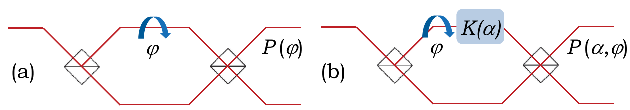

Fig. 1. (Color online) Optical configurations for (a) an angular mode analyzer and (b) a joint angular–radial mode analyzer.

$$E'(\rho,\theta;\varphi) = \int R(\theta,\theta';\varphi)E(\rho,\theta')\mathrm{d}\theta = E(\rho,\theta-\varphi;0), \quad (4)$$

in dyadic form in terms of $\{e^{i\ell\theta}\}$ as follows:

$$R(\theta,\theta';\varphi) = \delta(\theta-\theta'-\varphi) = \frac{1}{2\pi}\sum_{\ell} e^{-i\ell\varphi}e^{i\ell\theta}e^{-i\ell\theta'}. \quad (5)$$

We use this observation to construct a transformation that enables radial mode analysis. Consider modes having no angular variation (zero-OAM states), $u_{0p}(\rho) = A_{0p}L_p^{(0)}(\rho^2)e^{-\frac{1}{2}\rho^2}$, $A_{0p} = \frac{1}{\sqrt{\pi}}$, and beams that are superpositions of these modes, $E(\rho) = \sum_p c_{0p}u_{0p}(\rho)$. The question we pose is the following: how may one determine the weights $|c_{0p}|^2$, and, hence, the radial mode content?

In analogy with the spatial rotator, we construct an optical transformation that has the set of LG modes $\{u_{0p}(\rho)\}$ as eigenmodes and eigenvalues of the form $e^{ip\alpha}$, where the parameter α is the counterpart to the rotation angle φ. We expect α to vary periodically since the modal spectrum is discrete (indexed by p). In other words, we search for the optical transformation $K_0(\rho,\rho';\alpha)$ that has the property

$$\int K_0(\rho,\rho';\alpha)u_{0p}(\rho')\rho'\mathrm{d}\rho' = e^{ip\alpha}u_{0p}(\rho). \quad (6)$$

In dyadic form, the transformation K_0 takes the form

$$K_0(\rho,\rho';\alpha) = \sum_p e^{ip\alpha}u_{0p}(\rho)u^*_{0p}(\rho'). \quad (7)$$

This infinite sum of LG-mode products was identified by Namias in 1980 [8] as the fHT of fractional-order α and Bessel-order 0:

$$K_0(\rho,\rho';\alpha) = i\frac{e^{-ia}}{\sin a}\exp\left(-i\frac{\rho^2+\rho'^2}{2\tan a}\right)J_0\left(\frac{\rho\rho'}{\sin a}\right), \quad (8)$$

where $a = \frac{\pi}{2}\alpha$. The fHT has since been studied in optics [9] as the limiting form of the two-dimensional (2D) fractional Fourier transform [10,11] in systems having cylindrical symmetry. Thus, in order to analyze a cylindrically symmetric beam into the radial modes $\{u_{0p}(\rho)\}$, we construct a balanced MZI with an optical implementation of the fHT in one arm. The fractional-order α is swept and the output power $P(\alpha)$ is recorded, where

$$\begin{aligned} P(\alpha) &= 1 + \Re\int_0^\infty\int_0^{2\pi} E^*(\rho,\theta)\mathcal{F}_\alpha\{E(\rho,\theta)\}\rho\mathrm{d}\rho\mathrm{d}\theta \\ &= 1 + \sum_p |c_{0p}|^2\cos(p\alpha). \end{aligned} \quad (9)$$

Here, $\mathcal{F}_\alpha\{E(\rho,\theta)\} = \int K_0(\rho,\rho';\alpha)E(\rho',\theta)\rho'\mathrm{d}\rho'$. The FT of $P(\alpha)$ with respect to α yields the discrete spectrum $|c_{0p}|^2$. To the best of our knowledge, this is the first time that such an arrangement has been proposed.

We now turn to the *general* case of an optical beam having the decomposition in Eq. (2) that includes both radial *and* angular modal structure. The optical arrangement for analyzing the beam in terms of these modes is a balanced MZI with a spatial rotator in one arm and, in the other, an optical implementation of the fHT of fractional-order α, as shown in Fig. 1(b) (or, alternatively, placed in the same arm). The fHT of Bessel-order ℓ (which applies to beams containing $e^{i\ell\theta}$) is given by [8,9]

$$K_\ell(\rho,\rho';\alpha) = \frac{e^{i(1+\ell)\left(\frac{\pi}{2}-a\right)}}{\sin a}\exp\left(-i\frac{\rho^2+\rho'^2}{2\tan a}\right)J_\ell\left(\frac{\rho\rho'}{\sin a}\right). \quad (10)$$

This transformation has the properties

$$\int K_\ell(\rho,\rho';\alpha)u_{p\ell}(\rho',\theta)\rho'\mathrm{d}\rho' = e^{ip\alpha}u_{p\ell}(\rho,\theta), \quad (11)$$

$$K_\ell(\rho,\rho';\alpha) = \sum_p e^{ip\alpha}u_{\ell p}(\rho,\theta)u^*_{\ell p}(\rho',\theta). \quad (12)$$

After the spatial rotator *and* the fHT, the beam is

$$E'(\rho,\theta;\alpha,\varphi) = \sum_{\ell p} c_{p\ell}e^{i(\ell\varphi+p\alpha)}u_{\ell p}(\rho,\theta), \quad (13)$$

and the resulting 2D interferogram $P(\alpha,\varphi)$ is

$$\begin{aligned} P(\alpha,\varphi) &= 1 + \Re\int_0^\infty\int_0^{2\pi} E(\rho,\theta)E'^*(\rho,\theta;\alpha,\varphi)\rho\mathrm{d}\rho\mathrm{d}\theta \\ &= 1 + \sum_{\ell,p}|c_{\ell p}|^2\cos(p\alpha+\ell\varphi). \end{aligned} \quad (14)$$

The 2D FT of $P(\alpha,\varphi)$ with respect to α and φ, both of which change periodically, yields the discrete 2D modal landscape $|c_{\ell p}|^2$.

Several physical implementations of the optical fHT have been reported [12,13]. The simplest implementation consists of two spherical lenses where the fHT order is controlled by adjusting the two focal lengths and separation between the lenses [12]. A recent report demonstrates that two spatial light modulators with fixed separation may be used to implement the fHT with controllable order [13]. This implementation is particularly useful in our context since no moving parts are needed. The spatial rotation may be implemented using either a rotating dove prism or an arrangement of mirrors [14].

To elucidate the operation of this system, we examine several examples. First consider the case of a pure LG beam with $\ell = 1$ and $p = 2$, as shown in Fig. 2(a). The corresponding 2D interferogram $P(\alpha,\varphi)$, Eq. (14), is a pure sinusoid whose tilt in the (α,φ) plane determines the modal indices. Taking the FT of the interferogram reveals a single peak $|c_{1,2}|^2$ as expected. Our arrangement clearly distinguishes this beam from the pure LG beam having the same OAM $\ell = 1$ but different radial index $p = 4$ [Fig. 2(b)]. Next consider a multimode beam composed of a superposition of four modes with equal weights, each a product of an OAM and an LG mode, such that the radial and angular modal indices (ℓ, p) are correlated,

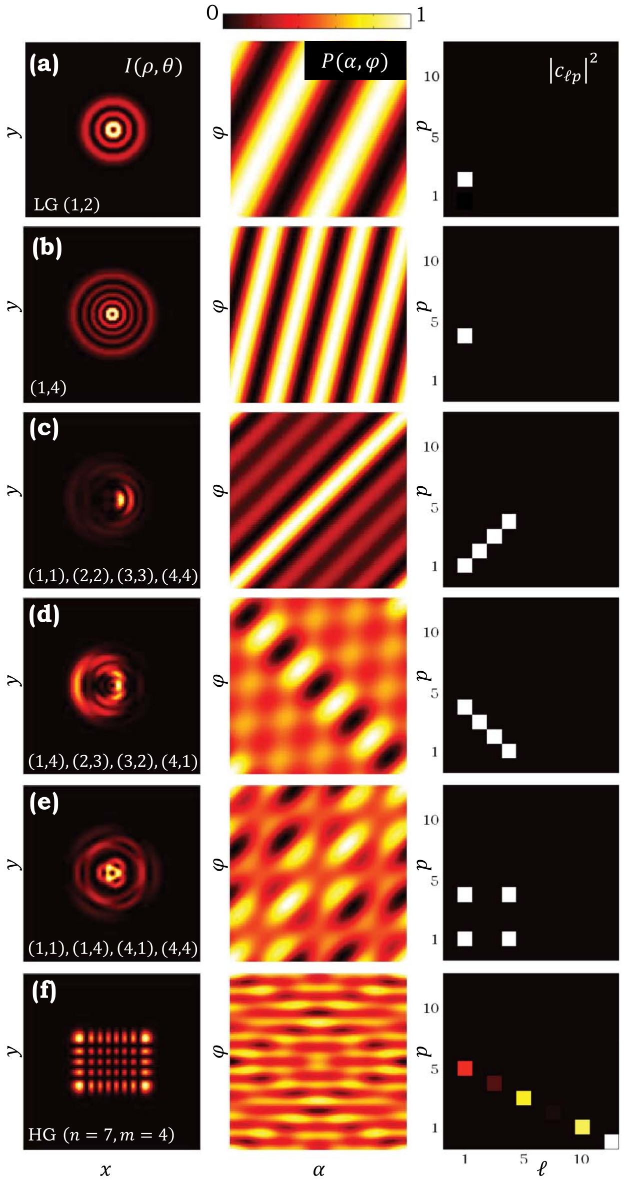

Fig. 2. (Color online) Examples of calculated 2D radial and angular spectra corresponding to the setup in Fig. 1(b). The first column displays the beam intensity $I(\rho,\theta) = |E(\rho,\theta)|^2$; each panel is 25.6×25.6 of the Gaussian mode standard deviation. The second column displays the 2D interferogram $P(\alpha,\varphi)$ and the third column displays the angular–radial modal spectrum $|c_{\ell p}|^2$, the 2D FT of $P(\alpha,\varphi)$. All panels are normalized to the peak value.

$\{(1,1),(2,2),(3,3),(4,4)\}$, as shown in Fig. 2(c). The correlation is visible in the 45° tilt in the fringes of $P(\alpha,\varphi)$, and its FT reveals the four superposed modes. Alternatively, a multimode beam that is a superposition of four equally weighted modes with anticorrelated (ℓ,p) indices, $\{(1,4),(2,3),(3,2),(4,1)\}$, is considered in Fig. 2(d). In this case, the interference fringes are tilted at −45°. Note that these two cases of correlated and anticorrelated angular–radial multimode beams would reveal the *same* angular spectrum at $\ell = 1, 2, 3, 4$, if only OAM were considered. Furthermore, if the two beams were subjected to a radial mode analyzer, the results once again would be identical for these two beams. Only measuring the radial and angular modal content simultaneously, using the arrangement in Fig. 1(b), for instance, uncovers this additional structure in the modal landscape and distinguishes the two beams. A superposition of modes with indices $\{(1,1),(1,4),(4,1),(4,4)\}$ [Fig. 2(e)] corresponds to a beam that is separable in the radial and angular coordinates as a result of the separability of the modal spectrum. Finally, a pure Hermite–Gaussian beam with indices $n = 7$ and $m = 4$ in the Cartesian coordinate system is analyzed in Fig. 2(f) and is resolved into a superposition of multiple OAM-LG modes with different weights.

In conclusion, we have presented an approach to measuring the full angular *and* radial spatial modal spectrum of a scalar optical beam based on a novel optical interferometer that combines a spatial rotator and an optical realization of the fHT. By varying the rotation angle and the order of the fHT, we obtain a 2D interferogram whose FT gives directly the desired modal spectrum.

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