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An electromagnetic world without polarization

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Abstract
The majority of natural sources (black-bodies, fluorescent bulbs, etc) generate completely un-polarized light; the majority of detectors (eyes, photo-cameras, photomultipliers, etc) are polarization-insensitive. To reflect this, we attempt to describe approximately electromagnetic waves without polarization. Corresponding scalar equations are non-trivial modifications of standard d'Alembert and Helmholtz equations to the case of spatially inhomogeneous propagation speed $v(\mathbf{r}) = 1/\sqrt{\varepsilon(\mathbf{r})\mu(\mathbf{r})}$. A description of Fresnel reflection (FR) and Goos–Hänchen shift for total internal reflection phenomena is given on the basis of these modified equations.

Keywords: polarization, wave propagation, transmission, absorption

(Some figures may appear in colour only in the online journal)

1. Introduction

Let us admit that we live in an un-polarized world in the following sense: (1) most sources of light in everyday life, e.g. the Sun with its black-body radiation, and incandescent and fluorescent light-bulbs, generate un-polarized light; (2) the detection of light, e.g. by eyes, by photo-cameras, or by photomultipliers, is mostly polarization-insensitive; (3) the propagation of light through clean air, water, glass, and other isotropic media is mostly un-affected by polarization.

All this gives us an incentive to look for an approximate description of electromagnetic waves in everyday life (which definitely are known to possess two transverse polarizations) via a scalar approach. We want to account for the refraction of light by the spatial gradients of propagation speed $v(\mathbf{r})$, be those gradients sharp (boundaries between two media) or smooth. Here

$$v(\mathbf{r})\ (\mathrm{m\ s^{-1}}) = \frac{1}{\sqrt{\varepsilon(\mathbf{r})\mu(\mathbf{r})}} = \frac{c}{n(\mathbf{r})},\qquad n(\mathbf{r}) = \sqrt{\frac{\varepsilon(\mathbf{r})\mu(\mathbf{r})}{\varepsilon_{\mathrm{vac}}\mu_{\mathrm{vac}}}}, \tag{1}$$

where $\varepsilon(\mathbf{r})$ (F m^{-1}) and $\mu(\mathbf{r})$ (H m^{-1}) are local dielectric permittivity and magnetic permeability, respectively; $n(\mathbf{r})$ is the local refractive index, and c, the speed of light in a vacuum.

Maxwell equations and material relationships are taken as follows:

$$\frac{\partial \mathbf{D}}{\partial t} = \mathrm{curl}\,\mathbf{H},\qquad \frac{\partial \mathbf{B}}{\partial t} = -\mathrm{curl}\,\mathbf{E}, \tag{2}$$

$$\mathbf{D} = \varepsilon(\mathbf{r})\mathbf{E},\qquad \mathbf{B} = \mu(\mathbf{r})\mathbf{H}. \tag{3}$$

Together they possess the property of being invariant under the substitutions

$$\mathbf{E} \Rightarrow \mathbf{H},\quad \mathbf{D} \Rightarrow \mathbf{B},\quad \mathbf{H} \Rightarrow -\mathbf{E},\quad \mathbf{B} \Rightarrow -\mathbf{D},\quad \varepsilon(\mathbf{r}) \Rightarrow \mu(\mathbf{r}),\quad \mu(\mathbf{r}) \Rightarrow \varepsilon(\mathbf{r}). \tag{4}$$

TE and TM have to be interchanged under the aforementioned substitution, and the influence of impedance upon them was not the same. (Reminder: TE and TM polarizations are introduced for complete classification of waves in layered media, i.e. in the media with properties depending on one Cartesian coordinate (e.g. on z) only. Meanwhile, the waves propagating in dielectric waveguides (with properties dependent on cylindrical coordinate $r = \sqrt{x^2 + y^2}$ only) are mostly 'hybrid', i.e. are neither of pure TE type nor of pure TM type.) Local propagation speed $v(\mathbf{r}) = 1/\sqrt{\varepsilon(\mathbf{r})\mu(\mathbf{r})}$ and local refractive index $n(\mathbf{r})$ stay invariant under transformation

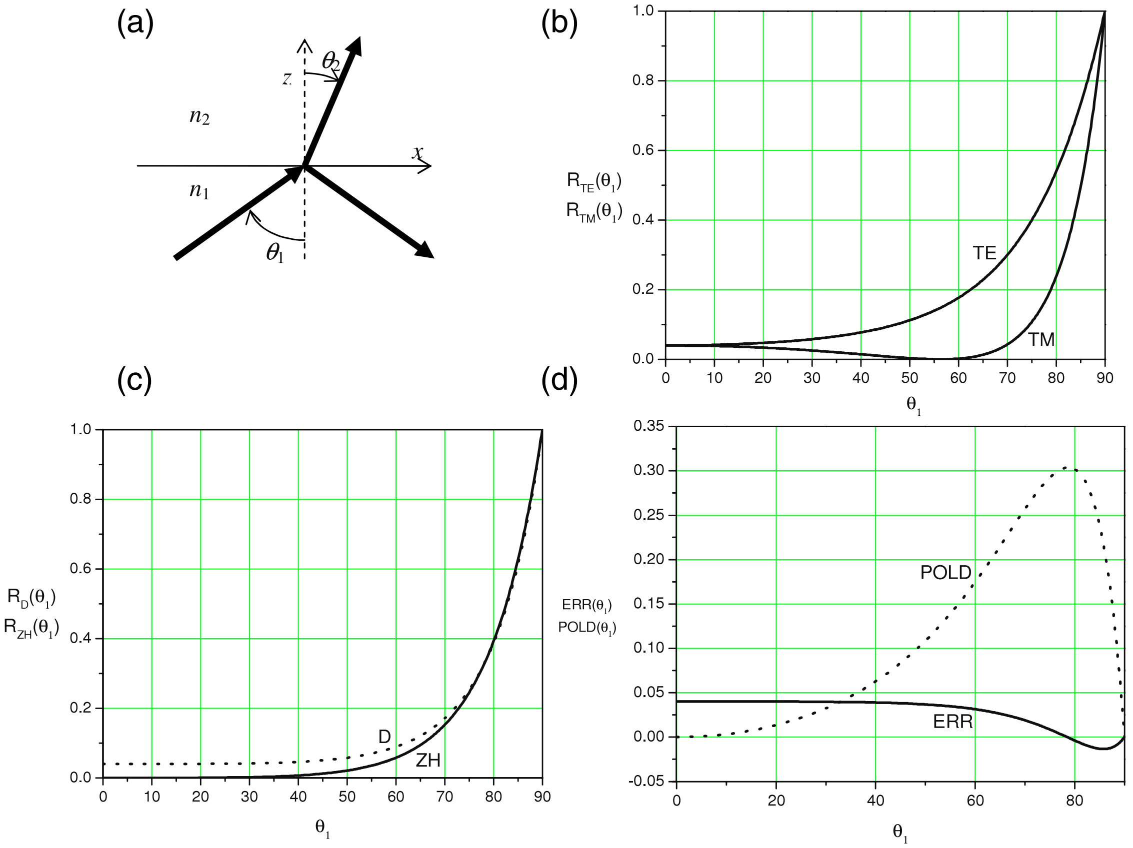

Figure 1. (a) Reflection and refraction by a flat boundary $z = 0$. (b) Fresnel reflection coefficients for intensity for TE (upper curve) and TM polarizations versus incidence angle θ_1 at the boundary between glass ($n_2 = 1.5$) and air ($n_1 = 1$). (c) Reflection coefficient in an 'un-polarized world', i.e. the arithmetic average of TE and TM reflectivities (dotted line) and the reflection coefficient according to the Z-Helmholtz equation (solid line.) (d) The 'error' of the Z-Helmholtz equation in the description of the un-polarized world (solid curve) and the difference between reflectivities of TE and TM polarizations (dotted curve).

(4). Meanwhile, the value of local impedance

$$Z(\mathbf{r})\ (\Omega) = \sqrt{\frac{\mu(\mathbf{r})}{\varepsilon(\mathbf{r})}} \tag{5}$$

changes under transformation TE $\Leftrightarrow$ TM given by equation (4):

$$Z_{\text{new}}(\mathbf{r}) = \sqrt{\frac{\mu_{\text{new}}(\mathbf{r})}{\varepsilon_{\text{new}}(\mathbf{r})}} = \frac{1}{Z_{\text{old}}(\mathbf{r})}. \tag{6}$$

In this work we explore the properties of the scalar waves (not the electromagnetic ones) in a medium, where the spatial profile of propagation speed $v(\mathbf{r})$ is considered inhomogeneous, but impedance is constant, $Z(\mathbf{r}) = Z_1 =$ const. After that we discuss which properties of light propagation in our everyday life can be approximately described by that scalar approach and which ones cannot.

Electromagnetic waves in media with constant impedance have been considered in a number of works of metamaterials and cloaks; see e.g. [1–3]. However, as for the media considered in numerous works on metamaterials, they have electromagnetic properties that are very different for two eigen-polarizations and thus they are out of the scope of interest of this paper.

A preliminary variant of this work was presented in [4].

2. Fresnel reflection in an un-polarized world and in the *Z*-Helmholtz approach

We consider a hypothesis here that equation (A.20) from the appendix for the complex scalar amplitude $p(\mathbf{r})$ of a monochromatic wave

$$Z = \text{const} \ \Rightarrow\ k(\mathbf{r})\vec{\nabla}\cdot\left\{\frac{1}{k(\mathbf{r})}\vec{\nabla}p(\mathbf{r})\right\} + k^2(\mathbf{r})p(\mathbf{r}) = 0, \tag{7}$$

(i.e. the *Z*-Helmholtz equation, in the terminology of the appendix) may serve the purpose of an approximate description of an 'un-polarized world'.

It is instructive (for subsequent comparison with optics) to calculate, within the framework of the *Z*-Helmholtz equation, the coefficient $R(\theta_1) = |r(\theta_1)|^2$ of Fresnel reflection for a plane wave, which is incident at an angle θ_1 to the normal of a sharp flat boundary between two media with $k(z) = k_1$ and $k(z) = k_2$, respectively, figure 1(a).

One looks for the solution

$$\begin{aligned} p_1(x, z) &= \exp(\mathrm{i}k_1 x\sin\theta_1)[1\exp(\mathrm{i}k_1 z\cos\theta_1) \\ &\quad + r\cdot\exp(-\mathrm{i}k_1 z\cos\theta_1)] \qquad \text{at } z < 0, \end{aligned} \tag{8}$$

$$\begin{aligned} p_2(x, z) &= \exp(\mathrm{i}k_2 x\sin\theta_2)[t\exp(\mathrm{i}k_2 z\cos\theta_2)] \\ &\quad \text{at } z > 0. \end{aligned} \tag{9}$$

Boundary conditions follow from equation (7) itself: continuity of p and of its normal (i.e. z)-derivative, divided by the corresponding value of k:

$$p_1(x, z = 0) = p_2(x, z = 0),$$
$$\frac{1}{k_1}\left(\frac{\partial p_1(x,z)}{\partial z}\right)_{z=0} = \frac{1}{k_2}\left(\frac{\partial p_2(x,z)}{\partial z}\right)_{z=0}. \qquad (10)$$

As a result, from x-dependence one gets Snell's law:

$$k_2 \sin\theta_2 = k_1 \sin\theta_1, \qquad (11)$$

and then

$$r = r_{\mathrm{ZH}}(\theta_1) = -\frac{B_{2,1}(\theta_1) - 1}{B_{2,1}(\theta_1) + 1}, \qquad (12)$$

where

$$B_{2,1}(\theta_1) = \frac{\cos\theta_2}{\cos\theta_1}, \qquad (13)$$

and the dependence of θ_2 on θ_1 is defined by Snell's law (11).

It is instructive to compare the (12), (13) with the well-known formulae for Fresnel reflection amplitudes for TE and TM polarizations reflected by a sharp boundary between two media with (generally different) k_1, Z_1 and k_2, Z_2.

Snell's law is still the same, equation (11), dependent on phase velocities $v_{1,2} = \omega/k_{1,2} \equiv c/n_{1,2}$ only. Standard electrodynamic boundary conditions of continuity of electric field components parallel to the boundary and of magnetic field components parallel to the boundary lead to well-known formulae, which may be found in textbooks on radiophysics:

$$r_{\mathrm{TE}}(\theta_1) \equiv r(E_y \leftarrow E_y) = \frac{[A_{2,1}/B_{2,1}(\theta_1)] - 1}{[A_{2,1}/B_{2,1}(\theta_1)] + 1}, \qquad (14)$$

$$r_{\mathrm{TM}}(\theta_1) \equiv r(E_x \leftarrow E_x) = \frac{[A_{2,1}B_{2,1}(\theta_1)] - 1}{[A_{2,1}B_{2,1}(\theta_1)] + 1}. \qquad (15)$$

Here we have introduced the ratio of impedances (which does not depend on incidence angle):

$$A_{2,1} = Z_2/Z_1. \qquad (16)$$

Figure 1(b) depicts well-known graphs of reflected intensities $R(\theta_1) \equiv |r(\theta_1)|^2$ for TE and TM polarizations, for a particular boundary between air ($n_1 = 1, Z_1 = 377\ \Omega$) and glass ($n_2 = 1.5, Z_2 = (377/n_2)\ \Omega$), parameter $A_{2,1} = 1/n_2 = 0.67$. At $\theta_1 = \arctan(n_2) \approx 56°$, polarization TM exhibits the Brewster phenomenon of zero reflection, i.e. 100% transmission. Normal incidence yields the well-known $R(0) = 4\%$ for both polarizations, while grazing incidence ($\theta_1 \to 90°$) yields almost 100% reflection, $R(90°) = 100\%$, again for both polarizations. Figure 1(c) depicts the reflection curve for the un-polarized world, i.e. arithmetic average: $\mathrm{RD}(\theta_1) = 0.5[R_{\mathrm{TE}}(\theta_1) + R_{\mathrm{TM}}(\theta_1)]$. Indeed, in the un-polarized world incident illumination is 50/50 split in intensities between TE and TM, and detection is equally sensitive for both polarizations. For comparison we depict on the same graph (figure 1(c)) the reflection curve calculated from the Z-Helmholtz equation, $R_{\mathrm{ZH}}(\theta_1)$. Figure 1(d) depicts the 'error' of the Z-Helmholtz description, $\mathrm{ERR}(\theta_1) = \mathrm{RD}(\theta_1) - R_{\mathrm{ZH}}(\theta_1)$, and the difference between two polarizations: $\mathrm{POLD}(\theta_1) = R_{\mathrm{TE}}(\theta_1) - R_{\mathrm{TM}}(\theta_1)$. One can see that the difference in reflectivities between the two polarizations is considerable: it reaches 30% at $\theta_1 = 79°$. Meanwhile the 'error' changes from +4% at normal incidence ($\theta_1 = 0$) to −1.3% at $\theta_1 = 86°$ and to zero at $\theta_1 = 90°$.

One can easily draw similar graphs and estimates for other values of the relative step of refractive index in the electrodynamics of non-magnetic media, for which one has impedance rigidly tied to the refractive index:

$$Z(\Omega) = \frac{377}{n} \qquad \text{(non-magnetic media)}. \qquad (17)$$

For example, for diamond-like material, $n_2 = 2.4$, the 'error' of the Z-Helmholtz approach is less than 17%, being at a maximum for normal incidence, while the difference between reflectivities of the two polarizations reaches 60%.

By the very sense of the approximation, the Z-Helmholtz equation predicts perfectly correct trajectories of rays, since phase velocities are taken true from the properties of the original media. As we have seen above, the Z-Helmholtz equation (7) (meaning Z = const) gives a quite realistic description of Fresnel-reflected intensities in the un-polarized world, at least for dielectric media with a not too large refractive index.

3. Goos–Hänchen shift

Another interesting application of the notion of the un-polarized world is the Goos–Hänchen shift of the centre of a beam reflected via total internal reflection (TIR), figure 2(a). The phenomenon of TIR may occur at the boundary of a medium '1', which has larger refractive index $n_1 = k_1 c/\omega$, with another medium, which has smaller refractive index $n_2 = k_2 c/\omega$, i.e. if $n_2 < n_1$. The condition of TIR is $k_2 < k_1 \sin(\theta_1)$, and it requires the angle of incidence θ_1 to be larger than the critical angle of TIR at that boundary. Goos–Hänchen shift is considered in a large number of publications and reviews, see e.g. [5–7].

The theory is based on the calculation of reflection coefficient, which happens to have a unit absolute value (in the case of TIR for non-absorbing media, real n_2 and n_1). Introducing tangential component $q \equiv k_{1x}$ of the wavevector $\mathbf{k_1}$ of the incident wave,

$$q = k_1 \sin(\theta_1), \qquad (18)$$

one can use the same Z-Helmholtz boundary conditions, equation (10), to get:

$$r(q) \equiv r(k_1 \sin(\theta_1)) = \frac{1 - \mathrm{i}\gamma}{1 + \mathrm{i}\gamma} \equiv \exp(-\mathrm{i}\delta) \;\Rightarrow$$
$$\delta = \delta(q) = 2\arctan[\gamma(q)], \qquad (19)$$

$$\gamma_{\mathrm{ZH}}(q) = \frac{\sqrt{(q/k_2)^2 - 1}}{\sqrt{1 - (q/k_1)^2}}. \qquad (20)$$

An approximation of the small angular spread of the incident beam allows us to derive a standard expression for the value of the Goos–Hänchen shift:

$$\Delta x = \frac{\mathrm{d}\delta(q)}{\mathrm{d}q}. \qquad (21)$$

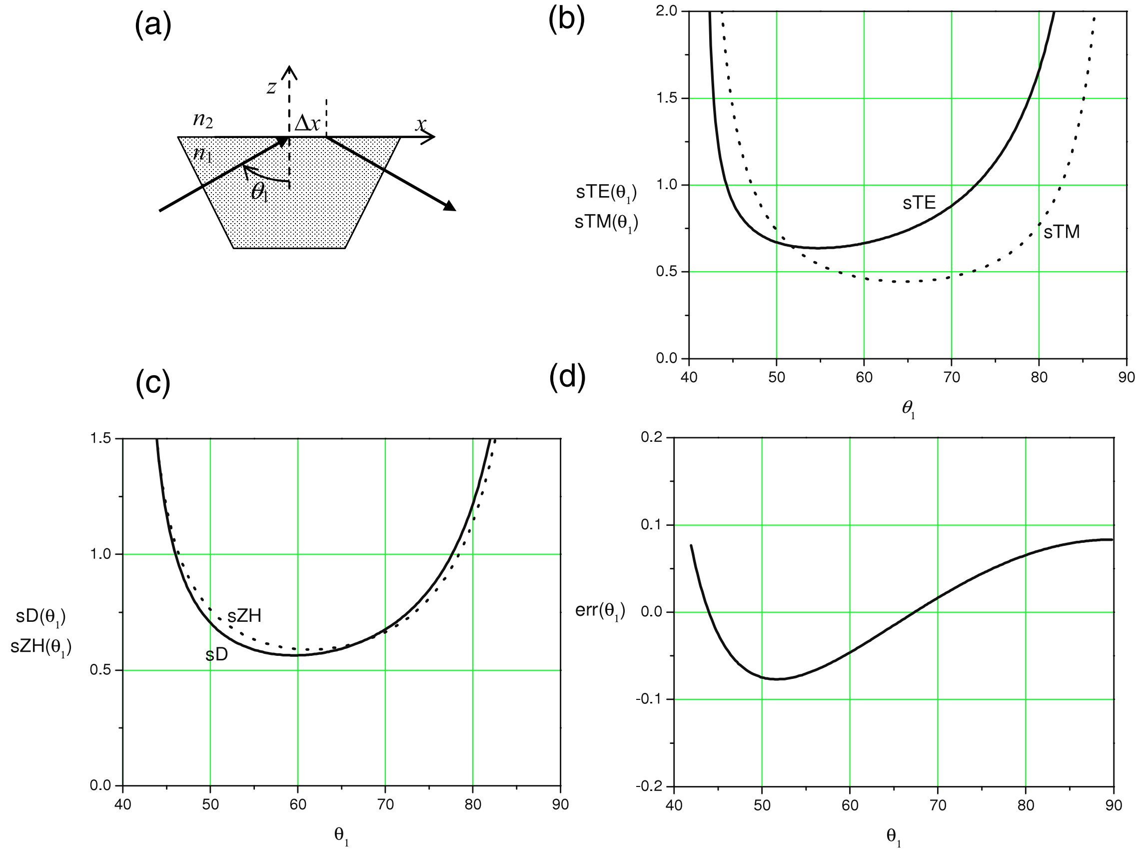

Figure 2. (a) Goos–Hänchen shift Δx at total internal reflection. (b) Goos–Hänchen shift Δx for TE-(sTE(θ_1), solid line) and TM polarizations (sTM(θ_1), dotted line), in units of vacuum wavelength, versus incidence angle θ_1 at the boundary between glass ($n_1 = 1.5$) and air ($n_2 = 1$). (c) Goos–Hänchen shift Δx in the un-polarized world, i.e. arithmetic average of TE- and TM-shifts (sD(θ_1), solid curve), and the value of the said shift according to the Z-Helmholtz equation (sZH(θ_1), dotted curve). (d) Relative error of the Z-Helmholtz equation in the description of the Goos–Hänchen shift Δx in the un-polarized world, err(θ_1) $= -1 + [\mathrm{sZH}(\theta_1)/\mathrm{sD}(\theta_1)]$, for the boundary between glass, $n_1 = 1.5$, and air, $n_2 = 1$.

One can also calculate well-known formulae for the Goos–Hänchen shift separately for TE and TM polarizations of light waves at the boundary of two media. The results are given by formulae (19), (21), but with a change of $\gamma_{\mathrm{ZH}}(q)$ from (20) to

$$\gamma_{\mathrm{TE}}(q) = \frac{1}{A_{2,1}}\gamma_{\mathrm{ZH}}(q),$$
$$\gamma_{\mathrm{TM}}(q) = A_{2,1}\gamma_{\mathrm{ZH}}(q), \qquad A_{2,1} = \frac{Z_2}{Z_1}, \tag{22}$$

respectively. The corresponding formulae are simply results of differentiation.

Figure 2(b) shows the graphs of dependence of shift for TIR at the boundary of glass ($n_1 = 1.5$) with air ($n_2 = 1$), separately for TE (sTE(θ_1)) and for TM (sTM(θ_1)) polarization; the value of the shifts is expressed in units of the vacuum wavelength. One can see a rather considerable difference between shifts for two polarizations. Figure 2(c) shows the arithmetic average of two shifts, (sD(θ_1)), which corresponds to the notion of the un-polarized world, and the shift calculated on the basis of the *Z*-Helmholtz approach, (sZH(θ_1)). Figure 2(d) depicts the relative error of the *Z*-Helmholtz approach, err(θ_1) $= -1 + [\mathrm{sZH}(\theta_1)/\mathrm{sD}(\theta_1)]$. One can see that for TIR at the glass–air boundary, the thus defined relative error does not exceed 10% at any angle, and mostly is less than 7%. For the boundary of a diamond-like medium ($n_1 = 2.4$) with air ($n_2 = 1$) the relative error does not exceed 30% at any angle and is mostly less than 15%.

From the very sense of the approximation, the *Z*-Helmholtz equation predicts perfectly correct trajectories of rays, since the phase velocities are taken true from the properties of the original media. One can see again that the *Z*-Helmholtz equation (meaning $Z = \mathrm{const}$) gives a quite realistic description of the Goos–Hänchen shift in the un-polarized world, at least for dielectric media with not too large refractive indices.

It is quite interesting also to consider angular shifts of the beam in the process of Fresnel reflection, as well as lateral shifts of circularly polarized beams (the Fedorov–Imbert effect). As for angular shift (see [8–10]), it exhibits very strong resonant behaviour in the vicinity of the Brewster angle, but for p-polarization only. Fedorov–Imbert shifts (see [9–15]) by their very nature have opposite sign for two circular polarizations. Thus both angular shifts and Fedorov–Imbert shifts constitute the phenomena, where the scalar approach of the *Z*-Helmholtz equation cannot give the correct answer, even an approximate one.

4. Strength of reflection and additivity of two causes of reflection

In the works [16–19] Zeldovich and Mokhov have introduced the notion of 'strength of reflection', a dimensionless parameter $S(\theta_1)$, defined through $r(\theta_1)$ by

$$r(\theta_1) = \tanh[S(\theta_1)] \equiv \frac{\exp[2S(\theta_1)] - 1}{\exp[2S(\theta_1)] + 1}. \tag{23}$$

One can see from equations (14) and (15) above that the values of Fresnel reflection strength satisfy simple laws of additivity (or, so to say 'subtractivity'):

$$\begin{aligned} S_{\mathrm{TE}}(\theta_1) &= S_{\mathrm{ZZ}} - S_{\mathrm{ZH}}(\theta_1), \\ S_{\mathrm{TM}}(\theta_1) &= S_{\mathrm{ZZ}} + S_{\mathrm{ZH}}(\theta_1). \end{aligned} \tag{24}$$

Here S_{ZZ} is the angle-independent strength of reflection, governed by a step of impedance only, and $S_{\mathrm{ZH}}(\theta_1)$ is the strength of reflection, calculated from the Z-Helmholtz equation and governed by step of propagation speed only:

$$\begin{aligned} S_{\mathrm{ZZ}} &= \frac{1}{2}\ln(A_{2,1}) \equiv \frac{1}{2}\ln\left(\frac{Z_2}{Z_1}\right), \\ S_{\mathrm{ZH}}(\theta_1) &= \frac{1}{2}\ln(B_{2,1}(\theta_1)) \equiv \frac{1}{2}\ln\left(\frac{\cos\theta_2}{\cos\theta_1}\right). \end{aligned} \tag{25}$$

Even for TIR, when $n_1 > n_2$ and $\cos(\theta_2)$ becomes purely imaginary one has

$$\begin{aligned} \text{TIR: } B_{2,1}(\theta_1) &\equiv \frac{\cos\theta_2}{\cos\theta_1} \\ &\equiv \mathrm{i}\gamma_{\mathrm{ZH}}(q = k_1\sin(\theta_1)) \equiv \mathrm{i}\frac{\sqrt{(q/k_2)^2 - 1}}{\sqrt{1-(q/k_1)^2}}, \end{aligned} \tag{26}$$

$$\begin{aligned} S_{\mathrm{TE}}(q) &= S_{\mathrm{ZH}}(q) + \frac{1}{2}\ln(A_{2,1}) \equiv -\frac{\mathrm{i}\pi}{4} \\ &\quad + \tfrac{1}{2}\ln(A_{2,1}) + \tfrac{1}{2}\ln(\gamma_{\mathrm{ZH}}(q)), \end{aligned} \tag{27}$$

$$\begin{aligned} S_{\mathrm{TE}}(q) &= S_{\mathrm{ZH}}(q) - \frac{1}{2}\ln(A_{2,1}) \equiv -\frac{\mathrm{i}\pi}{4} \\ &\quad - \tfrac{1}{2}\ln(A_{2,1}) + \tfrac{1}{2}\ln(\gamma_{\mathrm{ZH}}(q)). \end{aligned} \tag{28}$$

Thus we see that the notion of 'reflection strength S' helps in understanding many features of reflection. It was also shown in [18, 19] that reflection of longitudinal acoustic waves also satisfies the additivity law (24). The Brewster phenomenon, both in optics and in acoustics, may be interpreted as mutual cancellation of 'speed-step' (i.e. Z-Helmholtz) and of 'impedance-step' contributions to reflection strength.

5. Conclusion

The generalization of d'Alembert and Helmholtz equations for one scalar amplitude is given, which allows us to describe approximately optical phenomena in an un-polarized world. The approximation turned out to be surprisingly good. The notion of 'strength of reflection' is discussed in connection with these generalized equations.

Acknowledgments

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Appendix

In this appendix we derive the equations for longitudinal acoustic waves in a generally inhomogeneous liquid, where both the speed of sound $c(\mathbf{r})$ and background density $\rho_0(\mathbf{r})$ may be coordinate-dependent. Linearized equations of hydrodynamics (i.e. of acoustics) deal with the following dependent variables:

density

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) + \rho'(\mathbf{r}, t) + \rho''(\mathbf{r}, t)\ (\mathrm{kg\ m^{-3}}), \tag{A.1}$$

velocity

$$\mathbf{v}(\mathbf{r}, t) = \vec{0} + \mathbf{v}(\mathbf{r}, t)\ (\mathrm{m\ s^{-1}}), \tag{A.2}$$

and pressure

$$p(\mathbf{r}, t) = p_0 + p'(\mathbf{r}, t)\ (\mathrm{Pa} \equiv \mathrm{kg\ s^{-2}\ m^{-1}}). \tag{A.3}$$

Thus the background velocity is assumed to be absent in this stationary medium and background pressure is constant, $p_0 = \text{const}$.

The linearized Newton's second law may be written as

$$\rho_0(\mathbf{r})\frac{\partial\mathbf{v}(\mathbf{r}, t)}{\partial t} = -\vec{\nabla}p(\mathbf{r}, t). \tag{A.4}$$

The less evident part of the derivation is the separation of small variations of density due to the propagation of sound waves into two parts; $\rho(\mathbf{r}, t) - \rho_0(\mathbf{r}) = \rho'(\mathbf{r}, t) + \rho''(\mathbf{r}, t)$.

The first part, $\rho'(\mathbf{r}, t)$, describes the compressions and rarefactions of physically the same particles of acoustically moving liquid, that is to say the change of density in Lagrange coordinates.

The second one, $\rho''(\mathbf{r}, t)$, describes that part of local density change in Euler coordinates that arises due to the displacement of the particles of liquid from the previous position $(\mathbf{r} - \delta\mathbf{r})$, where the background density $\rho_0(\mathbf{r})$ was slightly different, so that

$$\begin{aligned} &\rho_0(\mathbf{r} - \delta\mathbf{r}) \approx \rho_0(\mathbf{r}) - \delta\mathbf{r}\cdot\vec{\nabla}\rho_0(\mathbf{r}), \\ &\delta\mathbf{r} = \mathbf{v}\delta t \ \Rightarrow\ \frac{\partial\rho''(\mathbf{r}, t)}{\partial t} = -\mathbf{v}(\mathbf{r}, t)\cdot\vec{\nabla}\rho_0(\mathbf{r}). \end{aligned} \tag{A.5}$$

The necessity for this separation of density variation into $\rho'(\mathbf{r}, t)$ and $\rho''(\mathbf{r}, t)$ is connected with the physical assumption about the material equation of state of compressible liquid. Namely, that the change in pressure is a result of the first (Lagrangian) part of density change only:

$$p'(\mathbf{r}, t) = c^2(\mathbf{r})\cdot\rho'(\mathbf{r}, t), \tag{A.6}$$

where

$$c^2(\mathbf{r}) \equiv \partial p/\partial\rho \tag{A.7}$$

is the square of the local speed of sound. Meanwhile, the mass conservation law may be written (in the same linearized approximation) as

$$\frac{\partial\left[\rho'(\mathbf{r},t)+\rho''(\mathbf{r},t)\right]}{\partial t}+\vec{\nabla}\cdot\{\mathbf{v}(\mathbf{r},t)\rho_0(\mathbf{r})\}=0. \quad (A.8)$$

As a result of very convenient cancellation, one can get rid of $\rho''(\mathbf{r},t)$ altogether:

$$\frac{\partial\rho'(\mathbf{r},t)}{\partial t}=-\rho_0(\mathbf{r})(\vec{\nabla}\cdot\mathbf{v}(\mathbf{r},t)). \quad (A.9)$$

Expressing $\rho'(\mathbf{r},t)$ via pressure variation by equation (A.6), one gets the system of coupled equations for velocity and pressure:

$$\begin{aligned}\frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t}&=-\frac{1}{\rho_0(\mathbf{r})}\vec{\nabla}p'(\mathbf{r},t),\\ \frac{\partial p'(\mathbf{r},t)}{\partial t}&=-\rho_0(\mathbf{r})c^2(\mathbf{r})(\vec{\nabla}\cdot\mathbf{v}(\mathbf{r},t)).\end{aligned} \quad (A.10)$$

It is convenient to introduce now the notion of the acoustic impedance $Z(\mathbf{r})$ of our liquid:

$$Z(\mathbf{r})=\rho_0(\mathbf{r})c(\mathbf{r})\equiv\rho_0(\mathbf{r})\sqrt{\partial p/\partial\rho}. \quad (A.11)$$

With this definition, our system may be reduced to

$$\frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t}=-\frac{c(\mathbf{r})}{Z(\mathbf{r})}\vec{\nabla}p(\mathbf{r},t), \quad (A.12)$$

$$\frac{\partial p(\mathbf{r},t)}{\partial t}=-Z(\mathbf{r})c(\mathbf{r})(\vec{\nabla}\cdot\mathbf{v}(\mathbf{r},t)). \quad (A.13)$$

The prime symbol in $p'(\mathbf{r},t)$ was omitted for brevity. Any serious textbook dealing with acoustics contains some equivalent of this derivation, see e.g. [20–22].

The consequence of this system (A.12), (A.13) is a second-order d'Alembert-type equation for pressure only:

$$\frac{1}{c^2(\mathbf{r})}\frac{\partial^2 p(\mathbf{r},t)}{\partial t^2}-\frac{Z(\mathbf{r})}{c(\mathbf{r})}\vec{\nabla}\cdot\left\{\frac{c(\mathbf{r})}{Z(\mathbf{r})}\vec{\nabla}p(\mathbf{r},t)\right\}=0. \quad (A.14)$$

It is instructive to consider two exceptional cases. The first one is when our liquid, having inhomogeneous speed of sound $c(\mathbf{r})$, possesses ideally constant impedance:

$$Z=\text{const}\ \Rightarrow\ \frac{1}{c^2(\mathbf{r})}\frac{\partial^2 p(\mathbf{r},t)}{\partial t^2}-\frac{1}{c(\mathbf{r})}\vec{\nabla}\cdot\{c(\mathbf{r})\vec{\nabla}p(\mathbf{r},t)\}=0. \quad (A.15)$$

The other exceptional case is when our liquid, having inhomogeneous impedance $Z(\mathbf{r})$, possesses ideally constant speed of sound:

$$c=\text{const}\ \Rightarrow\ \frac{1}{c^2}\frac{\partial^2 p(\mathbf{r},t)}{\partial t^2}-Z(\mathbf{r})\vec{\nabla}\cdot\left\{\frac{1}{Z(\mathbf{r})}\vec{\nabla}p(\mathbf{r},t)\right\}=0. \quad (A.16)$$

One can call equation (A.15) with $Z=\text{const}$ the Z-d'Alembert equation, while equation (A.16) with $c=\text{const}$ can be called the C-d'Alembert equation. Certainly, when $Z=\text{const}$ and $c=\text{const}$, these equations are both reduced to the standard d'Alembert equation:

$$c=\text{const},\qquad Z=\text{const}\ \Rightarrow\ \frac{1}{c^2}\frac{\partial^2 p(\mathbf{r},t)}{\partial t^2}-(\vec{\nabla}\cdot\vec{\nabla})p(\mathbf{r},t)=0. \quad (A.17)$$

Limiting oneself to monochromatic time-dependence $\propto \exp(-\mathrm{i}\omega t)$ and introducing the value of local wavevector

$$k(\mathbf{r})=\omega/c(\mathbf{r}), \quad (A.18)$$

one can get the equation for the general case:

$$Z(\mathbf{r})k(\mathbf{r})\vec{\nabla}\cdot\left\{\frac{1}{Z(\mathbf{r})k(\mathbf{r})}\vec{\nabla}p(\mathbf{r})\right\}+k^2(\mathbf{r})p(\mathbf{r})=0. \quad (A.19)$$

Its Z-Helmholtz variant becomes

$$Z=\text{const}\ \Rightarrow\ k(\mathbf{r})\vec{\nabla}\cdot\left\{\frac{1}{k(\mathbf{r})}\vec{\nabla}p(\mathbf{r})\right\}+k^2(\mathbf{r})p(\mathbf{r})=0, \quad (A.20)$$

while the C-Helmholtz (or K-Helmholtz) variant becomes

$$c=\text{const},\qquad k=(\omega/c)=\text{const}\ \Rightarrow Z(\mathbf{r})\vec{\nabla}\cdot\left\{\frac{1}{Z(\mathbf{r})}\vec{\nabla}p(\mathbf{r})\right\}+k^2p(\mathbf{r})=0, \quad (A.21)$$

and the standard Helmholtz equation is

$$Z=\text{const},\qquad k=(\omega/c)=\text{const}\ \Rightarrow\ (\vec{\nabla}\cdot\vec{\nabla})p(\mathbf{r})+k^2p(\mathbf{r})=0. \quad (A.22)$$

We use the Z-Helmholtz equation (A.20) in our consideration of the 'un-polarized world'.

Equation (A.20) may be obtained also as a consequence of the variational principle applied to the 'action functional I', taken in this form:

$$I[p(\mathbf{r}),(p(\mathbf{r}))^*]=\iiint\left[k(\mathbf{r})p(\mathbf{r})(p(\mathbf{r}))^*-\frac{1}{k(\mathbf{r})}\left(\vec{\nabla}p(\mathbf{r})\cdot(\vec{\nabla}p(\mathbf{r}))^*\right)\right]\mathrm{d}^3\mathbf{r} \quad (A.23)$$

with $k(\mathbf{r})=(k(\mathbf{r}))^*$. In the case of an arbitrary profile of $k(\mathbf{r})$, allowed to be complex-valued so that $k(\mathbf{r})=\mathrm{Re}(k(\mathbf{r}))+\mathrm{i}\,\mathrm{Im}(k(\mathbf{r}))$, the variational principle does not hold. Still, even in that complex case, any pair of solutions $p_1(\mathbf{r})$ and $p_2(\mathbf{r})$ of equation (A.20) can be arranged into Wronsky's vector, which has zero divergence as a consequence of equation (A.20):

$$\begin{gathered}\vec{W}_{1.2}(\mathbf{r})=\frac{1}{k(\mathbf{r})}(p_1(\mathbf{r})\vec{\nabla}p_2(\mathbf{r})-p_2(\mathbf{r})\vec{\nabla}p_1(\mathbf{r})),\\ \mathrm{div}(\vec{W}_{1.2}(\mathbf{r}))=0.\end{gathered} \quad (A.24)$$

Moreover, if $k(\mathbf{r})$ is real, then for any solution $p_1(\mathbf{r})$ of equation (A.20), the function $p_2(\mathbf{r})=(p_1(\mathbf{r}))^*$ is a solution of equation (A.20) as well, giving a time-reversed or 'phase-conjugate' solution, and thus for any solution $p(\mathbf{r})$ one gets:

$$\begin{gathered}\vec{j}(\mathbf{r})=\frac{1}{2\mathrm{i}k(\mathbf{r})}[(p(\mathbf{r}))^*\vec{\nabla}p(\mathbf{r})-p(\mathbf{r})\vec{\nabla}(p(\mathbf{r}))^*],\\ \mathrm{div}(\vec{j}(\mathbf{r}))=0.\end{gathered} \quad (A.25)$$

Thus the flux vector $\vec{j}(\mathbf{r})$ has zero divergence as a consequence of the Z-Helmholtz equation (A.20) with real $k(\mathbf{r})$. This may be considered as a manifestation of Noether's theorem whereby the invariance of functional (A.23) with respect to infinitesimally small transformation of phase,

$$\begin{aligned} &p(\mathbf{r}) \to p(\mathbf{r})(1 + \mathrm{i}\varepsilon), \\ &(p(\mathbf{r}))^* \to (p(\mathbf{r}))^*(1 - \mathrm{i}\varepsilon), \qquad \varepsilon \to 0. \end{aligned} \tag{A.26}$$

The one-dimensional case has an especially simple solution: for $k = k(z)$, two functions,

$$\begin{aligned} p_+(z) &= \exp\left[+\mathrm{i}\int_0^z k(z')\,\mathrm{d}z'\right], \\ p_-(z) &= \exp\left[-\mathrm{i}\int_0^z k(z')\,\mathrm{d}z'\right], \end{aligned} \tag{A.27}$$

happen to be exact solutions of equation (A.20) for arbitrary profile of $k(z)$, even a complex one. These functions look like WKB-type approximations without a pre-exponential factor, but they are exact solutions of equation (A.20)! This should not be a surprise as there is no reflection in 1D-propagation, if impedance is constant, either in acoustics or for radio-frequency coaxial cables or in electrodynamics in general, even for the step-like change of $k(z)$.

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