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Strength of electromagnetic, acoustic and Schrödinger reflections

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The notion of reflection strength S of a plane wave by an arbitrary non-absorbing layer is introduced, so that the intensity of reflection is $R=\tanh^2 S$. We have shown that the total strength of reflection by a sequence of elements is expressed through particular element strengths and mutual phases between them by a simple addition rule; in particular, its possible maximum is the sum of the absolute strengths of constituents. We show that the standard Fresnel reflection may be understood in terms of variable S as a sum or difference of two separate contributions, due to an impedance step and a speed step. Strength of reflection for propagating acoustic and quantum mechanical waves is also discussed.

Keywords: reflection; electromagnetic waves; acoustic waves; continuous spectrum; Schrödinger equation

1. Introduction

Reflection of light by layered media is the subject of an enormous number of works, including numerous monographs (see Brekhovskikh 1980; Haus 1984; Landau & Lifshitz 1984; Azzam & Bashara 1987; Yeh 1988; Born & Wolf 1999). In particular, reflection of light by volume Bragg gratings (VBGs) is usually studied in a slow-varying envelope approximation (see Kogelnik 1969; Collier *et al.* 1971; Zel'dovich *et al.* 1992) and recent experiments with VBGs in photo-thermo-refractive glass by Glebov *et al.* (2002). This work is devoted to the theoretical study of the general properties of reflecting elements. We allow for modulation of both dielectric susceptibility $\varepsilon(z)$ and magnetic permeability $\mu(z)$. The latter is especially important in connection with the new types of materials, including the ones with $\varepsilon<0$ and $\mu<0$ (see review by Pendry 2003).

2. Matrix method for description of strength of reflection

For a better perspective, let us first consider the transmission VBG, which couples two plane waves, A and B, both having a positive z-component of the Poynting vector: $P_z=|A|^2+|B|^2$. Here, the z-axis is normal to the boundaries of the VBG. Absence of absorption results in the conservation law: $P_z=\text{const}$.

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Writing the matrix relationship for wave coupling in linear media, $A(z)=N_{AA}\cdot A(0)+N_{AB}\cdot B(0)$ and $B(z)=N_{BA}\cdot A(0)+N_{BB}\cdot B(0)$ one comes to the conclusion that matrix $\hat{N}(z)$ must be unitary, i.e. it belongs to the elements of the unitary group U(2).

Consider now a reflecting device, where the waves A and B propagate in opposite directions with respect to the z-axis, so that $P_z=|A|^2-|B|^2$. Absence of absorption results in the conservation law: $|A|^2-|B|^2=\text{const}$. Writing the matrix relationship for wave coupling in linear media,

$$A(z) = M_{AA}\cdot A(0) + M_{AB}\cdot B(0) \quad \text{and} \quad B(z) = M_{BA}\cdot A(0) + M_{BB}\cdot B(0), \tag{2.1}$$

one can deduce from the assumption of energy conservation that the matrix $\hat{M}(z)$ satisfies the following conditions:

$$\hat{M} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad |\alpha|^2-|\gamma|^2 = 1, \qquad |\delta|^2-|\beta|^2 = 1 \quad \text{and} \quad \alpha\beta^* = \gamma\delta^*. \tag{2.2}$$

The most general form of such a matrix $\hat{M}$ depends on the following four real parameters: strength S, inessential phase ψ and two phases ζ and η,

$$\hat{M} = \mathrm{e}^{\mathrm{i}\psi}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\zeta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\zeta} \end{pmatrix}\begin{pmatrix} \cosh S & \sinh S \\ \sinh S & \cosh S \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\eta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\eta} \end{pmatrix}. \tag{2.3}$$

The determinant of such a matrix equals $\exp(2\mathrm{i}\psi)$, so that the modulus of that determinant is 1. Such matrices constitute a U(1, 1) group: their multiplication and taking inverse leave them within the same set. One can see an analogy between our transformation of wave amplitudes (2.1)–(2.3) and the Lorentz transformation if $|A|^2$ is playing the role of c^2t^2, $|B|^2$ the role of x^2 and the quantity $\tanh S$ corresponding to velocity parameter $\beta=V/c$, where V is the relative velocity of the coordinate frames.

Physical addition of two sequential elements with the parameters S_1, ψ_1, ζ_1, η_1 and S_2, ψ_2, ζ_2, η_2, respectively, yields the element described by the matrix $\hat{M}_3=\hat{M}_2\hat{M}_1$, i.e. the matrix of the same type as (2.3). Here is the expression for the resultant strength parameter S_3,

$$S_3 = \operatorname{arcsinh}\sqrt{\sinh^2(S_1+S_2)\cos^2\tau + \sinh^2(S_1-S_2)\sin^2\tau}, \quad \tau = \zeta_1-\eta_2, \tag{2.4}$$

which can vary due to a mutual phase difference between reflective elements.

The knowledge of the matrix $\hat{M}(z)$ allows one to find the amplitudes of the reflection and transmission coefficients. For example, for the problem with the wave A incident to the layer at the front, $z=0$, and with no wave B incident to the back, $z=L$, one substitutes boundary conditions $A(0)=1$ and $B(L)=0$ to get

$$0 = M_{BA}(L) + M_{BB}(L)\cdot r \Rightarrow r = r(B\leftarrow A) = -\frac{M_{BA}(L)}{M_{BB}(L)} = -\mathrm{e}^{-2\mathrm{i}\eta}\tanh S \tag{2.5}$$

and

$$R = |r(B\leftarrow A)|^2 = \tanh^2 S. \tag{2.6}$$

The presence of a hyperbolic tangent function is very satisfying: when the strength S goes to infinity, the reflection goes to 1 asymptotically. Kogelnik's (1969) theory of reflection by VBGs predicts the following value of the resultant strength:

$$\left.\begin{aligned} R_{\mathrm{VBG}} &= \tanh^2 S, \quad S = \operatorname{arcsinh}\left(S_0 \frac{\sinh\sqrt{S_0^2 - X^2}}{\sqrt{S_0^2 - X^2}}\right), \\ S_0 &= |\kappa| L, \quad X = \left(\frac{\omega n}{c}\cos\theta_{\mathrm{inside}} - \frac{Q}{2}\right) L. \end{aligned}\right\} \tag{2.7}$$

Here, S_0 is the strength of the VBG at perfect Bragg matching when the detuning parameter is $X=0$ and the coupling parameter $|\kappa|=1/2\ (n_1\omega/c)\cdot|\cos(\boldsymbol{E}_A, \boldsymbol{E}_B)|$ corresponds to modulation of the refractive index $\delta n(z)=n_1 \cos(Qz)$. The angle θ_{inside} is the propagation angle of the waves A and B inside the material of the VBG. Note that our formula (2.7) is mathematically identical to the result found by Kogelnik (1969), but is written in a somewhat different form.

3. Superposition law for strength values of several elements

If a reflective VBG slab has a certain residual reflection by the boundaries, $R_1 = |r_1|^2$ and $R_2 = |r_2|^2$, then the question arises about coherent interference between the main VBG reflection from equation (2.7) and these two extra contributions. Attentive consideration of the result (2.4) allows us to predict that, at any particular wavelength and/or angle of the incident wave, the strength S_{tot} of the total element will be within the limits

$$S_{\mathrm{VBG}} - |S_1| - |S_2| \le S_{\mathrm{tot}} \le S_{\mathrm{VBG}} + |S_1| + |S_2|, \qquad S_{1,2} = -\operatorname{arctanh} r_{1,2}. \tag{3.1}$$

Consider a particular example of the grating strength $S_{\mathrm{VBG}}=3.0$ at resonance, so that $R_{\mathrm{VBG}}=0.99$. Even if one has to deal with Fresnel reflections, $R_1=R_2=0.04$ for $n_0=1.5$, the modified reflection at the exact Bragg condition is within the boundaries $0.978 \le R_{\mathrm{tot}} \le 0.996$. On the contrary, in the spectral points of exactly zero R_{VBG}, the residual reflection varies within the interval

$$\tanh^2(S_1 - S_2) \le R \le \tanh^2(S_1 + S_2). \tag{3.2}$$

In particular, if $R_1=R_2=0.04$, then $0 \le R \le 0.148$. Another example is if $R_1 = R_2 = 0.003$, then $0 \le R \le 0.012$. Formula (3.2) allows us to also easily estimate maximum and minimum reflection of a Fabri–Perot interferometer with lossless mirrors of unequal reflectivities R_1 and R_2.

4. Understanding Fresnel reflection

Consider now a fundamental problem of electrodynamics: reflection of light by the sharp boundary between two media at the incidence angle θ_1, so that the refraction angle is θ_2. We denote by ε_1, μ_1, ε_2 and μ_2 the values of the dielectric permittivity and magnetic permeability in these two media, so that the values of phase propagation speed $v_{1,2}$ and impedance $Z_{1,2}$ are

$$v_j = \frac{c}{n_j}, \quad c = \frac{1}{\sqrt{\varepsilon_{\mathrm{vac}}\mu_{\mathrm{vac}}}}, \quad n_j = \sqrt{\frac{\varepsilon_j \mu_j}{\varepsilon_{\mathrm{vac}}\mu_{\mathrm{vac}}}} \quad \text{and} \quad Z_j = \sqrt{\frac{\mu_j}{\varepsilon_j}}, \quad j = 1, 2. \tag{4.1}$$

The angles θ_1 and θ_2 are related by Snell's law, which is governed by propagation speed ratio, i.e. by the ratio of refractive indices n_1 and n_2, namely $n_1 \sin \theta_1 = n_2 \sin \theta_2$. Cases of total internal reflection (TIR) and/or of absorbing second medium require the definition

$$\cos \theta_2 = \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1} = C_2' + \mathrm{i}C_2'', \quad C_2'' > 0. \tag{4.2}$$

The condition $C_2'' > 0$ guarantees the exponential decrease of the transmitted wave into the depth of the second medium. Amplitudes of reflection for transverse electric (TE) and transverse magnetic (TM) polarization are well known,

$$\left.\begin{aligned} r_{\mathrm{TE}} &\equiv r(E_y \leftarrow E_y) = \frac{\cos \theta_1/Z_1 - \cos \theta_2/Z_2}{\cos \theta_1/Z_1 + \cos \theta_2/Z_2} \\ \text{and} \quad r_{\mathrm{TM}} &\equiv r(E_x \leftarrow E_x) = -\frac{Z_1 \cos \theta_1 - Z_2 \cos \theta_2}{Z_1 \cos \theta_1 + Z_2 \cos \theta_2}. \end{aligned}\right\} \tag{4.3}$$

These expressions have two very instructive limiting cases. The first one is the case of two media that have the same phase speeds $v_1 = v_2$ (and thus refractive indices), so that $\theta_1 = \theta_2$. In a surprising manner, the reflection coefficients for such a problem do not depend on the angle and are equal to each other,

$$r_{\mathrm{TE}} = r_{\mathrm{TM}} \equiv r_{\Delta Z} = \frac{Z_2 - Z_1}{Z_2 + Z_1}. \tag{4.4}$$

The other case corresponds to media 1 and 2 having exactly the same impedances, $Z_1 = Z_2$, but different propagation speeds, i.e. $n_1 \neq n_2$. In this case, both reflection coefficients are equal to each other (up to the sign),

$$r_{\mathrm{TE}} = -r_{\mathrm{TM}} \equiv r_{\Delta v}(\theta_1) = \frac{\cos \theta_1 - \cos \theta_2}{\cos \theta_1 + \cos \theta_2}. \tag{4.5}$$

In particular, there is no reflection at normal incidence for the pair of impedance-matched media (stealth technology). Reflection strength values $S = -\operatorname{arctanh} r$ for these two limiting cases are

$$S_{\Delta Z} = \frac{1}{2} \ln\left(\frac{Z_1}{Z_2}\right) \quad \text{and} \quad S_{\Delta v}(\theta_1) = \frac{1}{2} \ln\left(\frac{\cos \theta_2}{\cos \theta_1}\right). \tag{4.6}$$

Here is a truly remarkable relationship, which we have found. One can produce the reflection strengths $S_{\mathrm{TE}}(\theta_1)$ and $S_{\mathrm{TM}}(\theta_1)$ by simple addition (for TE) or subtraction (for TM) of the speed-governed and impedance-governed contributions from (4.6),

$$S_{\mathrm{TE}}(\theta_1) = S_{\Delta Z} + S_{\Delta v}(\theta_1) \quad \text{and} \quad S_{\mathrm{TM}}(\theta_1) = S_{\Delta Z} - S_{\Delta v}(\theta_1), \tag{4.7}$$

and according to equation (2.6), $r = -\tanh S$. One can easily verify that the expressions (4.6) and (4.7) reproduce standard formulae (4.3) identically.

5. Maxwell equations for coupled waves: exact approach

We have actually found (4.7) for ourselves, not empirically, but have derived the result of additivity for reflection strength S directly from the Maxwell equations. The idea is to formulate *exact Maxwell equations* for the layered medium in terms of two coupled amplitudes A and B propagating with $P_z>0$ and $P_z<0$, respectively. We consider the incidence plane to be the (x, z)-plane, for a monochromatic wave $\propto \exp(-\mathrm{i}\omega t)$ incident upon a layered medium with the properties being z-dependent only. By θ_{air}, we denote the incidence angle of the wave in air, so that

$$\boldsymbol{k}_{\mathrm{air}} = \hat{\boldsymbol{x}} k_x + \hat{\boldsymbol{z}} k_{\mathrm{air},z}, \qquad k_x = \frac{\omega}{c} n_{\mathrm{air}} \sin\theta_{\mathrm{air}}, \qquad k_{\mathrm{air},z} = \frac{\omega}{c} n_{\mathrm{air}} \cos\theta_{\mathrm{air}}. \tag{5.1}$$

The waves in a layered medium are naturally separated into TE and TM parts. We will write electric and magnetic vectors of two polarizations through appropriately normalized components u_x, u_y, u_z and w_x, w_y, w_z, respectively,

$$\left.\begin{aligned} \mathrm{TE}: \quad \boldsymbol{E}(\boldsymbol{r}, t) &= -\hat{\boldsymbol{y}} u_y(z) \exp(\mathrm{i}k_x x - \mathrm{i}\omega t)\sqrt{Z(z)},\\ \boldsymbol{H}(\boldsymbol{r}, t) &= [\hat{\boldsymbol{x}} w_x(z) + \hat{\boldsymbol{z}} w_z(z)] \exp(\mathrm{i}k_x x - \mathrm{i}\omega t)/\sqrt{Z(z)}, \end{aligned}\right\} \tag{5.2}$$

and

$$\left.\begin{aligned} \mathrm{TM}: \quad \boldsymbol{E}(\boldsymbol{r}, t) &= [\hat{\boldsymbol{x}} u_x(z) + \hat{\boldsymbol{z}} u_z(z)] \exp(\mathrm{i}k_x x - \mathrm{i}\omega t)\sqrt{Z(z)},\\ \boldsymbol{H}(\boldsymbol{r}, t) &= \hat{\boldsymbol{y}} w_y(z) \exp(\mathrm{i}k_x x - \mathrm{i}\omega t)/\sqrt{Z(z)}. \end{aligned}\right\} \tag{5.3}$$

Here and below, we use quantities $k(z)$, $p(z)$, $g(z)$ and $f(z)$ defined by

$$k(z) = \frac{\omega n(z)}{c}, \qquad p(z) = \sqrt{k^2(z) - k_x^2} = k(z)\cos\theta(z), \tag{5.4}$$

$$g(z) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{1}{Z(z)} \quad \text{and} \quad f(z) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{p(z)}{k(z)} \equiv \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\ln\cos\theta(z). \tag{5.5}$$

The Maxwell equations for amplitudes of TE polarization are

$$\mathrm{i}ku_y = \partial_z w_x - \mathrm{i}k_x w_z + g w_x, \quad -\mathrm{i}kw_x = -\partial_z u_y + g u_y \quad \text{and} \quad -\mathrm{i}kw_z = \mathrm{i}k_x u_y, \tag{5.6}$$

and may be rewritten as

$$\partial_z u_y = g u_y + \mathrm{i}k w_x \quad \text{and} \quad \partial_z w_x = \mathrm{i}p^2/k u_y - g w_x. \tag{5.7}$$

It is convenient to introduce the amplitudes $A(z)$ and $B(z)$ for TE polarization by the definitions

$$\left.\begin{aligned} A_{\mathrm{TE}}(z)\exp(\mathrm{i}k_{\mathrm{air},z} z) &= \frac{1}{\sqrt{8}}\left(\sqrt{\frac{p}{k}} u_y(z) + \sqrt{\frac{k}{p}} w_x(z)\right)\\ \text{and} \qquad & \\ B_{\mathrm{TE}}(z)\exp(-\mathrm{i}k_{\mathrm{air},z} z) &= \frac{1}{\sqrt{8}}\left(\sqrt{\frac{p}{k}} u_y(z) - \sqrt{\frac{k}{p}} w_x(z)\right). \end{aligned}\right\} \tag{5.8}$$

The value of the z-component of the Poynting vector for any incidence angle at any point z is

$$P_z(z)=\frac{1}{4}\left(E_xH_y^*-E_yH_x^*+\text{c.c.}\right)=|A(z)|^2-|B(z)|^2. \tag{5.9}$$

It should be emphasized that we have deliberately chosen such normalization of amplitudes $A(z)$ and $B(z)$ so that relationship (5.9) is valid at any point z. One may consider the transformation (5.8) as a transition to 'slow-varying envelopes' $A(z)$ and $B(z)$. It is important to emphasize, however, that no approximations were made up to this point. Indeed, the exact Maxwell equations for TE polarization are reduced to a very simple coupled pair,

$$\left.\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}A_{\mathrm{TE}}(z)\\ B_{\mathrm{TE}}(z)\end{pmatrix}=\hat{V}_{\mathrm{TE}}\begin{pmatrix}A_{\mathrm{TE}}(z)\\ B_{\mathrm{TE}}(z)\end{pmatrix}\\ &\text{and}\\ &\hat{V}_{\mathrm{TE}}=\begin{pmatrix}\mathrm{i}(p(z)-k_{\mathrm{air},z}) & (g(z)+f(z))\exp(-2\mathrm{i}k_{\mathrm{air},z}z)\\ (g(z)+f(z))\exp(2\mathrm{i}k_{\mathrm{air},z}z) & -\mathrm{i}(p(z)-k_{\mathrm{air},z})\end{pmatrix}.\end{aligned}\right\} \tag{5.10}$$

A similar set of transformations may be done for TM polarization

$$\left.\begin{aligned}&-\mathrm{i}ku_x=-\partial_z w_y-gw_y,\quad -\mathrm{i}ku_z=\mathrm{i}k_xw_y,\\ &-\mathrm{i}kw_y=\mathrm{i}k_xu_z-\partial_z u_x+gu_x\Leftrightarrow\partial_z u_x=gu_x+\mathrm{i}p^2/k\quad\text{and}\quad\partial_z w_y=\mathrm{i}ku_x-gw_y,\end{aligned}\right\} \tag{5.11}$$

with the same parameters $k(z)$, $g(z)$ and $p(z)$. The amplitudes of coupled TM waves are

$$\left.\begin{aligned}&A_{\mathrm{TM}}(z)\exp(\mathrm{i}k_{\mathrm{air},z}z)=\frac{1}{\sqrt{8}}\left(\sqrt{\frac{k}{p}}u_x(z)+\sqrt{\frac{p}{k}}w_y(z)\right)\\ &\text{and}\\ &B_{\mathrm{TM}}(z)\exp(-\mathrm{i}k_{\mathrm{air},z}z)=\frac{1}{\sqrt{8}}\left(\sqrt{\frac{k}{p}}u_x(z)-\sqrt{\frac{p}{k}}w_y(z)\right).\end{aligned}\right\} \tag{5.12}$$

Finally, the exact Maxwell equations for TM polarization are

$$\left.\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}A_{\mathrm{TM}}(z)\\ B_{\mathrm{TM}}(z)\end{pmatrix}=\hat{V}_{\mathrm{TM}}\begin{pmatrix}A_{\mathrm{TM}}(z)\\ B_{\mathrm{TM}}(z)\end{pmatrix}\\ &\text{and}\\ &\hat{V}_{\mathrm{TM}}=\begin{pmatrix}\mathrm{i}(p(z)-k_{\mathrm{air},z}) & (g(z)-f(z))\exp(-2\mathrm{i}k_{\mathrm{air},z}z)\\ (g(z)-f(z))\exp(2\mathrm{i}k_{\mathrm{air},z}z) & -\mathrm{i}(p(z)-k_{\mathrm{air},z})\end{pmatrix},\end{aligned}\right\} \tag{5.13}$$

with the same parameters $f(z)$ and $g(z)$ as in (5.5). The gradient functions $f(z)$ (related to propagation speed) and $g(z)$ (related to impedance) enter as a sum (for TE polarization) or as a difference (for TM polarization) into our 'coupled' equations. Sharp steps of $n(z)$ and $Z(z)$ yield our result: equations (4.7).

The notion of reflection strength was originally introduced by us for non-absorbing media. It is worth noting that the reflection by a sharp step with an absorbing second medium and a reflection in the TIR regime are both described by $S = -\operatorname{arctanh}(r)$ and the equations (4.6) and (4.7) are still valid. In particular, the TIR regime corresponds to

$$S_{\Delta Z} = \frac{1}{2}\ln\left(\frac{Z_1}{Z_2}\right) \quad \text{and} \quad S_{\Delta v}(\theta_1) = \mathrm{i}\frac{\pi}{4} + \frac{1}{2}\ln\left(\frac{\sqrt{(n_1/n_2)^2\sin^2\theta_1 - 1}}{\cos\theta_1}\right). \tag{5.14}$$

As expected, $|r| = |\tanh(\mathrm{i}\pi/4 + \operatorname{Re} S)| = 1$ for the case of TIR.

6. Acoustic and Schrödinger reflections

It is interesting to consider the reflection of longitudinal acoustic waves from the boundary between two liquids that have densities ρ_1 and ρ_2, propagation speeds c_1 and c_2 and therefore acoustic impedances $Z_1 = \rho_1 c_1$ and $Z_2 = \rho_2 c_2$, respectively. A well-known expression for the reflection coefficient for the wave's pressure (Brekhovskikh 1980; Landau & Lifshitz 1987) is

$$r_{\text{longitud}} \equiv r(p \leftarrow p) = \frac{\cos\theta_1/Z_1 - \cos\theta_2/Z_2}{\cos\theta_1/Z_1 + \cos\theta_2/Z_2}. \tag{6.1}$$

For this acoustic case, we see that again the reflection strength is given by the sum of two contributions,

$$r_{\text{longitud}} = -\tanh[S_p(\theta_1)] \quad \text{and} \quad S_p(\theta_1) = S_{\Delta Z} + S_{\Delta c}(\theta_1). \tag{6.2}$$

The Schrödinger equation for the motion of an electron in a given Bloch band should generally account for two kinds of spatial inhomogeneity (see Nelin 2007). One of them is $U(\boldsymbol{r})$ (Joule), i.e. the spatial profile of the bottom of the Brillouin zone. The other one must describe $m(\boldsymbol{r})$ (kg), i.e. the inhomogeneity of the coefficient $1/(2m)$ in the parabolic approximation $E(\boldsymbol{p}) = p^2/(2m)$ of the dependence of electron energy in the vicinity of the bottom of the Brillouin zone on the momentum $\boldsymbol{p}$. The corresponding Hermitian Hamilton operator is

$$\hat{H} = \hat{\boldsymbol{p}}\frac{1}{2m(\boldsymbol{r})}\hat{\boldsymbol{p}} + U(\boldsymbol{r}) \tag{6.3}$$

and it acts upon the wavefunction ψ. Consider now the motion with fixed energy E, i.e. $\psi(\boldsymbol{r},t) = \psi(\boldsymbol{r})\exp(-\mathrm{i}Et/\hbar)$. Then, the stationary Schrödinger equation takes the form

$$m(\boldsymbol{r})\nabla\left[\frac{1}{m(\boldsymbol{r})}\nabla\psi\right] + \frac{2m(\boldsymbol{r})}{\hbar^2}[E - U(\boldsymbol{r})]\psi = 0. \tag{6.4}$$

If $m(\boldsymbol{r}) = \text{const.}$, then equation (6.4) is reduced to the conventional Schrödinger equation. It is convenient to introduce the following two quantities: the 'kinematic parameter' $k(\boldsymbol{r})$, i.e. the wavenumber, and the 'dynamical parameter' $Z(\boldsymbol{r})$, the analogue of impedance, by the definitions

$$\left.\begin{aligned} k(\boldsymbol{r}) &= \frac{1}{\hbar}\sqrt{2m(\boldsymbol{r})[E - U(\boldsymbol{r})]}, \quad [k] = \mathrm{m}^{-1} \\ \text{and} \quad Z(\boldsymbol{r}) &= \frac{m(\boldsymbol{r})}{\hbar k(\boldsymbol{r})} = \sqrt{\frac{m(\boldsymbol{r})}{2[E - U(\boldsymbol{r})]}}, \quad [Z] = \mathrm{s\,m}^{-1}. \end{aligned}\right\} \tag{6.5}$$

Numerically, the parameter $Z(\boldsymbol{r})$ coincides with the local value of the inverse group velocity. With these notations, the equation (6.4) takes the form

$$Z(\boldsymbol{r})k(\boldsymbol{r})\nabla\left[\frac{1}{Z(\boldsymbol{r})k(\boldsymbol{r})}\nabla\psi\right]+k^2(\boldsymbol{r})\psi=0. \tag{6.6}$$

This equation (6.6) has two interesting limiting cases. One of them is equation (6.6) with $Z=Z_0=\text{const.}$,

$$k(\boldsymbol{r})\nabla\left[\frac{1}{k(\boldsymbol{r})}\nabla\psi\right]+k^2(\boldsymbol{r})\psi=0, \tag{6.7}$$

and we may call it the Z-Helmholtz equation, to emphasize the condition $Z=Z_0=\text{const}$. The other limiting case is when $k=k_0=\text{const.}$, but the impedance is coordinate dependent,

$$Z(\boldsymbol{r})\nabla\left[\frac{1}{Z(\boldsymbol{r})}\nabla\psi\right]+k_0^2\psi=0, \tag{6.8}$$

and for similar reasons (6.8) may be labelled as the k-Helmholtz equation. Finally, when both $Z=Z_0=\text{const.}$ and $k=k_0=\text{const.}$, we come to the standard Helmholtz equation $\nabla^2\psi+k_0^2\psi=0$. The usual (i.e. with $\hbar kZ=m_0=\text{const.}$) stationary Schrödinger equation $\nabla^2\psi+k^2(\boldsymbol{r})\psi=0$ is a certain intermediate case between Z- and k-Helmholtz equations.

The flux $\boldsymbol{J}$ (particles/(m^2s)) for a plane mono-energetic wave $\psi=\exp(-\mathrm{i}\boldsymbol{kr})$ in the homogeneous part of the medium equals $\boldsymbol{J}=(\boldsymbol{k}/k)v|\psi|^2=(\boldsymbol{k}/k)|\psi|^2/Z$. The conservation law, which is valid as a consequence of the mono-energetic Schrödinger equation (6.6), is

$$\operatorname{div}\boldsymbol{J}(\boldsymbol{r},t)=0,\quad \boldsymbol{J}(\boldsymbol{r},t)=\frac{\mathrm{i}\hbar}{2m(\boldsymbol{r})}(\psi\nabla\psi^*-\psi^*\nabla\psi). \tag{6.9}$$

The problem of reflection for the one-dimensional stationary Schrödinger equation,

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}+k^2(z)\psi(z)=0,\qquad k^2(z)=\frac{2m}{\hbar^2}(E-V(z)), \tag{6.10}$$

may also be solved by the coupled waves approach. Namely, we will assume for definiteness that $k^2(z)>0$ and introduce local amplitudes $A(z)$ and $B(z)$ by

$$\left.\begin{aligned} A(z)\mathrm{e}^{\mathrm{i}k_0z}&=\sqrt{k(z)}\psi-\frac{\mathrm{i}}{\sqrt{k(z)}}\frac{\mathrm{d}\psi}{\mathrm{d}z}\\ \text{and}\quad B(z)\mathrm{e}^{-\mathrm{i}k_0z}&=\sqrt{k(z)}\psi+\frac{\mathrm{i}}{\sqrt{k(z)}}\frac{\mathrm{d}\psi}{\mathrm{d}z},\qquad k_0=\frac{\sqrt{2mE}}{\hbar}.\end{aligned}\right\} \tag{6.11}$$

The advantage of amplitudes $A(z)$ and $B(z)$ is that the flux $J_z(z)$ is expressed very simply,

$$J(z)=\frac{\mathrm{i}\hbar}{2m}\left(\psi\frac{\mathrm{d}\psi^*}{\mathrm{d}z}-\psi^*\frac{\mathrm{d}\psi}{\mathrm{d}z}\right)=\frac{\hbar}{4m}\left(|A(z)|^2-|B(z)|^2\right). \tag{6.12}$$

This flux is conserved, $J(z)=\text{const.}$, as a consequence of (6.4) with real mass and potential. The uniqueness of the representation (6.11) is guaranteed if one requires that, in the homogenous part of the medium, our waves A and B do not interact with each other. It is important that equation (6.10) is exactly

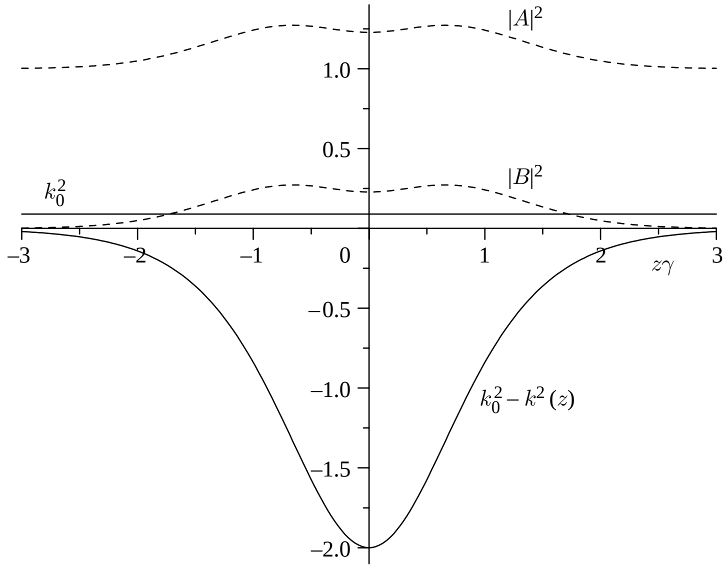

Figure 1. An example of the profiles of $|A(z)|^2$ and $|B(z)|^2$ normalized to $P(z)=1$ for the problem with $k^2(z)=k_0^2+\alpha^2 s(s+1)/\cosh^2(\alpha z)$. Solid lines, potential well $k_0^2-k^2(z)=-2/\cosh^2 z$ yielding non-reflection and incident beam energy $k_0^2=0.09$. Dashed lines, fluxes $|A(z)|^2$ and $|B(z)|^2$ of counter-propagating wave function components.

equivalent to the system of coupled first-order equations,

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} A(z) \\ B(z) \end{pmatrix} = \begin{pmatrix} \mathrm{i}(k(z)-k_0) & F(z)\mathrm{e}^{-2\mathrm{i}k_0 z} \\ F(z)\mathrm{e}^{2\mathrm{i}k_0 z} & -\mathrm{i}(k(z)-k_0) \end{pmatrix}\begin{pmatrix} A(z) \\ B(z) \end{pmatrix}, \quad F(z)=\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}z}\ln k^2(z). \tag{6.13}$$

Numerical (or analytical, whenever possible) solution of this *exact* system in the form of the $\hat{M}$-matrix allows us to find amplitudes of reflection and transmission. It should be emphasized that boundary conditions for system (6.13) are applied only at one end, e.g. at $z=-\infty$, so that one should solve the Cauchy problem, for which any standard code of integration of ordinary differential equations works very well. Figure 1 shows an example of the profiles of $|A(z)|^2$ and $|B(z)|^2$ normalized to $P(z)=1$ for the problem with

$$k^2(z) = k_0^2 + \frac{\alpha^2 s(s+1)}{\cosh^2(\alpha z)}, \tag{6.14}$$

at the particular 'non-reflective' value $s=1$ at $k_0/\alpha=0.3$ (see Landau & Lifshitz 1981).

Reflection of a wave for tilted incidence by the sharp boundary between two media with different values, (k_1, Z_1) and (k_2, Z_2), requires the analogue of Snell's law, $k_1 \sin\theta_1 = k_2 \sin\theta_2$; here θ_1 and θ_2 are the angles of the momentum with the normal to the boundary in the respective media. Boundary conditions of continuity of the wave function ψ and of $(\partial\psi/\partial Z)/(Zk)$ yield the following expression for the reflection coefficient:

$$r=-\tanh S, \quad S=S_{\Delta k}+S_{\Delta Z}, \quad S_{\Delta k}=\frac{1}{2}\ln\left(\frac{\cos\theta_2}{\cos\theta_1}\right), \quad S_{\Delta Z}=\frac{1}{2}\ln\left(\frac{Z_1}{Z_2}\right). \tag{6.15}$$

7. Conclusion

To conclude, the notion of reflection strength S is introduced, so that the intensity of reflection is $R = \tanh^2 S$. The strength of total reflection S_{tot} by a sequence of lossless elements is expressed through particular element strengths and mutual phases between them by the simple addition rule (2.4); in particular, its possible maximum is the sum of the absolute strengths of the constituents. We have shown that the amplitudes of the standard processes of Fresnel reflection may be understood in terms of S as a linear sum or difference of the following two independent contributions: the impedance step and the speed step. A similar result is obtained for the reflection of longitudinal acoustic waves. The one-dimensional Schrödinger equation is also treated with specially introduced amplitudes of coupled counter-propagating components.

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