Laser Phys. Lett. 12 (2015) 015004 (6pp) doi:10.1088/1612-2011/12/1/015004

Quality deterioration of self-phase modulated Gaussian beams

Sergiy Mokhov, Leonid Glebov and Boris Zeldovich

CREOL – the College of Optics and Photonics, University of Central Florida, Orlando, FL 32816, USA

E-mail: smokhov@creol.ucf.edu

Received 31 July 2014
Accepted for publication 30 October 2014
Published 27 November 2014

Abstract

The beam quality parameter M_x^2 for self-phase modulated Gaussian and super-Gaussian beams has been calculated analytically. These results can be used for experimental measurements of nonlinearities in optical materials. A super-Gaussian beam with a size comparable to a Gaussian one demonstrates much more stable behavior with propagation through Kerr media. The deteriorated beam quality parameter has also been calculated analytically in the case of higher order localized radial phase modes. The presented analysis of beam quality degradation due to localized phase aberrations is more efficient than the standard one based on polynomial aperture aberrations.

Keywords: beam quality, M2, self-phase modulation, aberration, Kerr nonlinearity, super-Gaussian

(Some figures may appear in colour only in the online journal)

1. Introduction

Optical beams of finite size diverge with propagation in free space due to their wave nature. With diffraction of the plane wavefront at an aperture, the linear divergence is determined by the corresponding angle in the far-field zone, proportional to the ratio of wavelength to aperture size. As a result, the product of aperture size by divergence angle divided by wavelength remains the same for a particular aperture shape, e.g. a cylindrical one, independent of aperture size.

Similarly one can characterize the diffraction properties of propagating localized aperture-free beams by the product of the minimal observed beam size times the divergence angle divided by the wavelength according to Siegman [1]. This dimensionless parameter should depend only on the inner structure of the beam amplitude profile. One such propagation-invariant parameter describing the beam divergence in one transverse x-direction is defined as

$$M_x{}^2 = 2k\sqrt{\langle x^2\rangle_{\min}}\sqrt{\langle\theta_x^2\rangle}\,. \quad (1)$$

Here $k = 2\pi n_0/\lambda$ is the wave vector with vacuum wavelength λ in linear lossless medium with refractive index n_0. Second-order moments of the size and angle are calculated under the assumption of zero first-order moments $\langle x\rangle$ and $\langle\theta_x\rangle$.

The parameter $M_x{}^2$ is widely used for beam quality characterization in the laser science and industry. Its theoretical properties are discussed from different points of view in [2–5]. Probably the simplest way to prove the invariance of $M_x{}^2$ with propagation is based on explicit paraxial propagation equations for second-order moments. In this framework the second-order moments of beam profile complex amplitude known at some propagation position can be presented in explicit form. Corresponding expressions are provided in further equations (5) and (8). With the use of these equations we were able to derive the beam quality of the self-phase modulated Gaussian beam analytically. The main result of this letter is presented in equation (15). It describes the physical effect of quality deterioration of the beam propagated through a thin layer of Kerr media with arbitrarily strong nonlinearity. Problems of similar types of localized self beam distortion and resulting quality deterioration occur in the analysis of high-power laser generation and operation [6–8].

Wavefront phase aberrations are traditionally studied in problems related to imaging systems and they are conventionally represented as polynomial expansions in terms of Zernike polynomials over an aperture of unit radius [9]. After introduction of the beam quality parameter, Siegman analyzed the influence of spherical aberration on it in [10]. The effect of different Zernike aberrations on beam quality was studied in particular in [11–14]. We discuss here the applicability limitations of a polynomial aberration approach to beam quality characterization of self-phase modulated beams.

2. Propagation equations for second-order moments

The Helmholtz equation for electric field amplitude $E(\vec{r}, z)$ in the case of a propagating coherent laser beam along the z-direction is reduced to the scalar paraxial wave equation (PWE) for slow z-varying complex amplitude $U(\vec{r}, z)$, where we denoted the transverse coordinates (x, y) by $\vec{r}$:

$$E(\vec{r}, z) = U(\vec{r}, z) \cdot e^{ikz}, \quad \frac{\partial U(\vec{r}, z)}{\partial z} - \frac{i}{2k} \nabla^2_{\vec{r}} U(\vec{r}, z) = 0. \quad (2)$$

If the transverse profile $U(\vec{r}, z_1)$ is known at some propagation position $z = z_1$, then this boundary condition determines the solution of equation (2) at any z through the Huygens integral in the Fresnel approximation [15]. In the case of a normalized Gaussian amplitude profile $U_{G0}(\vec{r}, 0) = (2/\pi)^{1/2}/w_0 \cdot \exp(-r^2/w_0^2)$ with size w_0 at $z = 0$, this well-known propagation solution of the PWE is:

$$U_G(\vec{r}, z) = \frac{-ik}{2\pi z} \int U_G(\vec{r}', 0) e^{\frac{ik(\vec{r} - \vec{r}')^2}{2z}} dS' = \frac{\sqrt{\frac{2}{\pi}}\, e^{-\frac{r^2/w_0^2}{1 + iz/z_R}}}{w_0 \left(1 + i\frac{z}{z_R}\right)},$$

$$z_R = \frac{1}{2} k w_0^2, \quad \langle x^2 \rangle_G = \frac{\int x^2 |U_G|^2 \, dS}{\int |U_G|^2 \, dS = 1} = \frac{w_0^2}{4}\left(1 + \frac{z^2}{z_R^2}\right). \quad (3)$$

Here z_R is the so-called Rayleigh length, where the square of the Gaussian beam size increases by two times.

The preservation of total power $P = \int |U|^2 dS$ of an arbitrary localized beam with continuous profile can be easily proven by utilizing the PWE (2) first for $\partial U^*/\partial z$:

$$\frac{\partial P}{\partial z} = \int \left(\frac{1}{2ik} \nabla^2_{\vec{r}} U^* \cdot U + U^* \frac{\partial U}{\partial z} \right) dS \equiv 0. \quad (4)$$

The last identity is obtained by applying integration by parts for the first term with Laplacian followed by combining a zero-factor corresponding to the PWE (2) for U.

The propagation equations for the second-order moments starting from $\langle x^2 \rangle = P^{-1} \int x^2 |U|^2 \, dS$ can be derived by applying the PWE and integrating by parts similarly to equation (4), and these equations have the following explicit form:

$$\frac{\partial}{\partial z} \langle x^2 \rangle = \frac{i}{kP} \int x \left(\frac{\partial U^*}{\partial x} U - U^* \frac{\partial U}{\partial x} \right) dS = 2 \langle x\theta_x \rangle,$$

$$\frac{\partial}{\partial z} \langle x\theta_x \rangle = \frac{1}{k^2 P} \int \left| \frac{\partial U}{\partial x} \right|^2 dS = \langle \theta_x^2 \rangle, \quad \frac{\partial}{\partial z} \langle \theta_x^2 \rangle = 0. \quad (5)$$

In these expressions the angular variable has the operator representation $\hat{\theta}_x = -i/k \cdot \partial/\partial x$ in coordinate space.

Moments of angular variables have a clear interpretation after transferring to angular space $\vec{\theta} = (\theta_x, \theta_y)$:

$$\langle \theta_x^2 \rangle = \frac{1}{P} \int |C|^2 \theta_x^2 d\Omega, \quad C(\vec{\theta}, z) = \frac{k}{2\pi} \int U(\vec{r}, z) e^{-ik\vec{r} \cdot \vec{\theta}} dS. \quad (6)$$

The same beam power can be calculated from the angular content distribution $P = \int |C|^2 d\Omega$, $d\Omega = d\theta_x d\theta_y$. The mixed moment $\langle x\theta_x \rangle$ has a less obvious physical interpretation and is related to the wavefront curvature radius $R_x = \langle x^2 \rangle / \langle x\theta_x \rangle$.

3. Explicit definition of M_x^2 and its properties

In order to compare the intrinsic diffraction features of different beam profiles, the second-order coordinate-angular moments should be calculated for each beam in corresponding coordinate frames where the first-order moments $\langle x \rangle$ and $\langle \theta_x \rangle$ are equal to zero. Otherwise substitutions $\langle x^2 \rangle \to \langle x^2 \rangle - \langle x \rangle^2$, $\langle \theta_x^2 \rangle \to \langle \theta_x^2 \rangle - \langle \theta_x \rangle^2$, $\langle x\theta_x \rangle \to \langle x\theta_x \rangle - \langle x \rangle \langle \theta_x \rangle$ should be applied.

From equation (5) it is easy to check the propagation invariance of the following parameter:

$$\frac{1}{4k^2} \frac{\partial}{\partial z} (M_x^2)^2 \equiv 0, \quad M_x^2 = 2k \sqrt{\langle \theta_x^2 \rangle \langle x^2 \rangle - \langle x\theta_x \rangle^2}. \quad (7)$$

This invariant can be calculated from the complex amplitude of the propagating beam $U(\vec{r}, z)$ at any z according to explicit expressions for the second-order moments $\langle \theta_x^2 \rangle$ and $\langle x\theta_x \rangle$ provided in equation (5). If complex amplitude is presented through real functions of amplitude profile $\rho(\vec{r})$ and phase $\Phi(\vec{r})$ then expressions (5) are equivalent to

$$U = \rho(\vec{r}) e^{i\Phi(\vec{r})}, \quad \langle \theta_x^2 \rangle = \frac{1}{k^2 P} \int \left[\left(\frac{\partial \rho}{\partial x} \right)^2 + \rho^2 \left(\frac{\partial \Phi}{\partial x} \right)^2 \right] dS,$$

$$\langle x\theta_x \rangle = k^{-1} P^{-1} \int x \rho^2 \partial\Phi / \partial x \, dS, \quad \langle x^2 \rangle = P^{-1} \int x^2 \rho^2 dS,$$

$$\langle x \rangle = P^{-1} \int x \rho^2 dS, \quad \langle \theta_x \rangle = k^{-1} P^{-1} \int \rho^2 \partial\Phi / \partial x \, dS. \quad (8)$$

The propagation solution of a system of differential equation (5) for $\langle x^2 \rangle (z)$ can be obtained in terms of the moments calculated at some particular $z = z_1$. Then the position z_{min} of the beam waist along the x-axis related to the minimal value of $\langle x^2 \rangle$ can be introduced correspondingly

$$\langle x^2 \rangle (z) = \langle x^2 \rangle_{min} + \langle \theta_x^2 \rangle (z - z_{min})^2,$$

$$z_{min} = z_1 - \langle x\theta_x \rangle_1 / \langle \theta_x^2 \rangle, \quad \langle x^2 \rangle_{min} = M_x^4 / \left(4k^2 \langle \theta_x^2 \rangle\right). \quad (9)$$

From these, one can see the equivalence of definitions of M_x^2 presented in equations (1) and (7). Quadratic fitting of the experimental dependence $\langle x^2 \rangle (z)$ gives $\langle x^2 \rangle_{min}$ and $\langle \theta_x^2 \rangle$, thus defining M_x^2.

According to equation (9), a beam has its waist in the x-direction at such position z_1 where $\langle x\theta_x \rangle_1 = 0$. In particular this occurs when the amplitude profile at this position is a real function, $\Phi = 0$, so that the moment is equal to zero according to equation (8).

In the case of a Gaussian beam, comparison of equations (3) and (9) gives quickly a unity value for M_x^2:

$$\langle x_G^2 \rangle_{min} = \frac{1}{4} w_0^2, \quad \langle \theta_{x,G}^2 \rangle = k^{-2} w_0^{-2}, \quad M_{x,G}^2 = 1. \quad (10)$$

If for an arbitrary beam we consider a corresponding 'underlying' Gaussian beam with the same size $\langle x^2\rangle_{\min}$ then according to equations (9) and (10) we obtain another equivalent definition of M_x^2 as the ratio of divergent angles in the x-direction of the studied beam and a Gaussian beam of equal waist sizes:

$$M_x^2 = \sqrt{\langle\theta_x^2\rangle \,/\, \left\langle\theta_{x,G}^2\right\rangle}_{\text{at } \langle x_G^2\rangle_{\min} = \langle x^2\rangle_{\min}}. \quad (11)$$

The value $M_x^2 = 1$ in equation (10) for a Gaussian beam profile is the minimum achievable. This fact follows from the non-negative identity for arbitrary real numbers a and b:

$$0 \le 4\langle\theta_x^2\rangle \int |(kx + (b - \mathrm{i}a)/k \cdot \partial/\partial x)\, U|^2 \,\mathrm{d}S \,/\, \int |U|^2 \,\mathrm{d}S \equiv$$
$$\equiv \left(2\langle\theta_x^2\rangle a + 2k\langle x\theta_x\rangle\right)^2 + \left(2\langle\theta_x^2\rangle b - 1\right)^2 + M_x^4 - 1. \quad (12)$$

Values of a and b satisfying the zero equality define the corresponding first-order partial differential equation for amplitude U according to the first line of (12) and its solution has a factorized Gaussian dependence of U on x.

In [16] the authors found a propagation invariant parameter based on second moments of both x- and y-coordinates, which is also invariant under rotation of the coordinate system in the (x, y) plane transverse to the propagation z-direction with natural assumption of zero first-order moments:

$$M_{2D}^4 = \tfrac{1}{2}M_x^4 + \tfrac{1}{2}M_y^4 + 4k^2\left(\langle xy\rangle \left\langle\theta_x\theta_y\right\rangle - \left\langle x\theta_y\right\rangle \langle y\theta_x\rangle\right) \quad (13)$$

It does not change with arbitrary astigmatic quadratic aberration and has a minimal unity value for astigmatic Gaussian beam profiles. If the amplitude profile is symmetric with respect to the x- or y-axes then the last term in brackets is equal to zero; otherwise cross-moments are calculated similarly to equation (5). For example $\left\langle\theta_x\theta_y\right\rangle$ has $\partial U^*/\partial x\cdot\partial U/\partial y$ under the integral sign. Astigmatic beams are usually characterized by separate values M_x^2 and M_y^2.

To date, we presented how the beam propagation invariant M_x^2 can be defined and explicitly calculated from an arbitrary transverse amplitude profile $U(\vec{r}, z_1)$. Some facts about M_x^2 presented above were discussed previously in other terms by other people. Below we use the introduced theoretical results in our notations for further original analysis of beam quality of self-phase modulated Gaussian beams.

4. M_x^2 of a self-phase modulated Gaussian beam

Now let us discuss the deterioration of M_x^2 of a Gaussian beam after phase aberrations. One of those is the self-phase modulation occurring upon propagation in a medium with an intensity-dependent refractive index $n = n_0 + n_2 I$. If the propagation length Δz inside such a medium placed at Gaussian beam waist position $z = 0$ is much smaller than the Rayleigh length from equation (3) then, after propagation, the initial real Gaussian profile $\rho(r)$ gains the phase distortion $\Phi(r)$ proportional to the intensity profile $I(r) = \rho^2(r)$

$$U_{\mathrm{Gd}}(r, z = 0) = \rho(r)\mathrm{e}^{\mathrm{i}\Phi(r)}, \quad \rho(r) = \sqrt{I(0)}\,\mathrm{e}^{-r^2/w_0^2},$$
$$\Phi(r) = \Phi_0 \mathrm{e}^{-2r^2/w_0^2}, \quad \Phi_0 = n_2 I(0) 2\pi/\lambda\cdot\Delta z. \quad (14)$$

Here Φ_0 is the maximum value of the phase at the center of the beam; this is the single parameter describing the problem.

Due to axial symmetry, each moment in equation (8) equals half of the corresponding one calculated for variable r. This is similar to the classical mechanics calculation of inertia moment I_x of a thin rotating body along the x-axis perpendicular to the z-axis of rotation $I_x = \frac{1}{2}I_z = \pi\int_0^\infty r^2\mu(r) r\mathrm{d}r$, where $\mu(r)$ is the areal mass density. In this way, integrations in equation (8) for the distorted radial Gaussian profile $U_{\mathrm{Gd}}(r)$ from equation (14) give analytical expressions

$$P = 2\pi\int_0^\infty \rho^2 r\mathrm{d}r, \quad \langle x^2\rangle = \frac{\pi}{P}\int_0^\infty r^2\rho^2 r\mathrm{d}r = \frac{w_0^2}{4},$$
$$\langle x\theta_x\rangle = \frac{\pi}{kP}\int_0^\infty r\rho^2\frac{\mathrm{d}\Phi}{\mathrm{d}r} r\mathrm{d}r = -\frac{\Phi_0}{4k},$$
$$\langle\theta_x^2\rangle = \frac{\pi}{k^2P}\int_0^\infty\left[\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)^2 + \rho^2\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\right] r\mathrm{d}r = \frac{1 + \frac{4}{9}\Phi_0^2}{k^2w_0^2},$$
$$\rightarrow \quad M_{x,\mathrm{G}}^2 = 2k\sqrt{\langle x^2\rangle\langle\theta_x^2\rangle - \langle x\theta_x\rangle^2} = \sqrt{1 + 7/36\cdot\Phi_0^2}. \quad (15)$$

As we should expect the obtained result does not depend on the dimensional beam size w_0.

The analytical expression for M_x^2 in equation (15) of a self-phase-modulated Gaussian beam profile from equation (14) is the main result of this paper. The same result occurs for a divergent Gaussian beam propagating a small distance $\Delta z \ll z_{\mathrm{R}}$ in a nonlinear medium at an arbitrary position $z \ne 0$ from its waist, see equation (3), because initial incident profile $U_{\mathrm{G}}(r,z)$ has a certain Gouy phase and quadratic divergence phase, which do not affect the final M_x^2. Thus it depends on the central phase value $\Phi_{r=0}(z)$ proportional to the Kerr nonlinearity coefficient n_2 and the central intensity $I(r = 0,z) = 2P/(\pi w^2(z))$ at z-position in the same manner of equation (15). For example, if the thin Kerr element is placed in the Gaussian laser beam at position $z = z_{\mathrm{R}}$ then $\Phi_0(z_{\mathrm{R}})$ is twice smaller than Φ_0 at the beam waist position because $w^2(z_{\mathrm{R}}) = 2w_0^2$ according to equation (3).

The obtained result could be applied for determination of n_2 in some materials. The M_x^2 can be measured with the use of well-established experimental techniques, and the central phase Φ_0 of self-phase modulation due to the optical nonlinearity will be expressed from equation (15). After that, knowing the intensity at the center of the beam $I(0)$, the corresponding n_2 can be found according to equation (14). The basic technique for determining n_2, the so-called 'Z-scan' [17], is also based on measurements of the changed propagation properties of a Gaussian beam passed through the nonlinear material. We propose here though to analyze these changes in terms of M_x^2.

The beam quality parameter is often discussed with long distance optical power delivery. With a fixed size output aperture it is important that the outgoing beam has a reasonably small angular divergence with M_x^2 not strongly exceeding one. Many approaches for increasing the output laser power have been studied including increasing the power of single laser sources followed by incoherent combining of separate laser beams in the same angular direction through a common aperture [18, 19]. At the last combining stage the common aperture elements operate at high total combined laser power and due to unavoidable material absorption thermal aberrations may play a significant role in degradation of the combined performance. These thermal aberrations are similar to self-phase modulation processes in nonlinear optics described by equation (14) and lead to beam quality deterioration similar to the one in equation (15).

Usually, due to transverse heat dissipation, in a particular optical element and/or its varying thickness, the localized aberration profile $\Phi(r)$ is different from the intensity profile for self-phase modulation in equation (14). In order to still get an analytical expression for M_x^2, the arbitrary localized profile $\Phi(r)$ can be decomposed into higher-order modes related to the intensity Gaussian profile $I(r)$. Using the three lowest orthogonal radial Laguerre–Gaussian modes $p_n(r)$ we get from the integrals in equation (15) the following:

$$\Phi(r) = \sum_n \Phi_n p_n(r) \ , \quad p_n(r) = L_n(4r^2/w_0^2)\mathrm{e}^{-2r^2/w_0^2},$$
$$M_x^4 = 1 + \frac{7\Phi_0^2}{36} + \frac{53\Phi_0\Phi_1}{54} + \frac{149\Phi_1^2}{108} + \frac{8\Phi_0\Phi_2}{9} + \frac{248\Phi_1\Phi_2}{81} + \frac{556\Phi_2^2}{243} + \dots, \tag{16}$$

where Φ_n are phase values of each radial mode contributing to the total phase $\Phi(0)$ at the center of the beam as $p_n(0) = 1$.

Generalization of analytical formulae (15), (16) in cases of arbitrary self-phase modulated profiles will lead to cumbersome expressions. In the general case of arbitrary complex amplitude of beam profile $U(\vec{r})$, the beam quality M_x^2 can be calculated by numerical integrations in equation (5) or the equivalent equation (8) for factorized real amplitude and phase. For profiles with azimuthal symmetry we presented equation (15) with 1D radial integration, which could be performed analytically or numerically. If the radial mode content of the profile with azimuthal symmetry $U(r)$ is known then M_x^2 is determined algebraically by expansion coefficients of radial Laguerre–Gaussian modes [20]

$$U(r) = \sum_n \alpha_n \rho_n(r) \ , \quad \rho_n = \frac{(-1)^n}{w_0}\sqrt{\frac{2}{\pi}}L_n\left(\frac{2r^2}{w_0^2}\right)\mathrm{e}^{-\frac{r^2}{w_0^2}},$$
$$2\pi\int_0^\infty \rho_n\rho_m r\,\mathrm{d}r = \delta_{nm}, \quad P = \sum_n |\alpha_n|^2,$$
$$M_x^4P^2 = \left(\sum_n (2n+1)|\alpha_n|^2\right)^2 - 4\left|\sum_n n\alpha_n^*\alpha_{n-1}\right|^2. \tag{17}$$

Here modes $\rho_n(r)$ are presented in real form at waist position $z = 0$ in equation (3) for illustration purposes but actually coefficients α_n can be used for a set of modes at any z-position.

Similar expressions for M_x^2 of arbitrary profile $U(\vec{r})$ without azimuthal symmetry can be utilized in particular through known expansion coefficients of orthonormal Hermite–Gaussian basis

$$U = \sum_n g_n(y)\psi_n(x) \ , \quad \psi_n = \frac{\sqrt[4]{2/\pi}}{\sqrt{2^n n!\,w_0}}H_n\left(\frac{\sqrt{2}x}{w_0}\right)\mathrm{e}^{-\frac{x^2}{w_0^2}},$$
$$q_{mn} = \int g_m^* g_n \mathrm{d}y = \sum_k c_{mk}^* c_{nk}, \quad g_n = \sum_k c_{nk}\psi_k(y),$$
$$M_x^4P^2 = \left(\sum_n (2n+1)q_{nn}\right)^2 - 4\left|\sum_n \sqrt{n(n-1)}\,q_{n,n-2}\right|^2, \tag{18}$$

where the total beam power is $P = \sum_n q_{nn}$. One can derive the result (18) for the beam quality straightforward from equation (5) with the use of recurrent relations for $x\psi_n$ and $\mathrm{d}\psi_n/\mathrm{d}x$. Set of involved reference modes again can be used at any z not just $z = 0$ as ψ_n in equation (18).

5. Polynomial aberration analysis of self-phase modulation

We would like to demonstrate that polynomial aberration approaches cannot be efficient for problems with localized aberrations caused by the propagating beam itself. First, the self-phase modulation $\Phi(r) = \exp(-2r^2/w_0^2)$ in equation (14) can be considered as a limit of corresponding Taylor series $\Phi_N(r) = \Phi_0\Sigma_{m=0..N}(-2r^2/w_0^2)^m/m!$ at N going to infinity. One can check that for profiles $\rho(r)\exp(\mathrm{i}\Phi_N(r))$ similar to equation (14) the sequence of analytical values of M_x^2 calculated similar to equation (15) is eventually divergent with an increase of N. Thus straightforward polynomial expansion of self-phase modulation is not applicable for analytic analysis of beam quality deterioration.

Now let us take into consideration an aperture of unit radius for referring the Gaussian beam profile to it. Suppose the Gaussian beam going through this aperture has relative dimensionless beam size $w_0 = 0.4$ in order to avoid significant power loss by aperturing. We are considering dimensionless spacial values as the dependence of M_x^2 only on the amplitude profile shape but not the actual size as was mentioned before. Let us take into account the particular value $\Phi_0 = 1$ rad of self-phase modulation which degrades the beam quality to $M_x^2 = 1.093$ according to equation (15). The corresponding aberration profile $\Phi(r)$ from equation (14) can be fitted by radial polynomials $\Psi_N(r)$ of increasing even power N, which is equivalent to using Zernike radial aberrations of up to Nth order. Figure 1(*a*) demonstrates $\Phi(r)$ with some of these minimal-square error fits $\Psi_N(r)$. Figure 1(*b*) shows how the M_x^2 calculated for the incident beam profile from equation (14) with improving phase fits $\Psi_N(r)$ converges to the proper value with increasing N. The corresponding integrals in equation (15) are calculated within the aperture range $r \le 1$ where the phase polynomials $\Psi_N(r)$ have been used to fit $\Phi(r)$.

We see that for the relatively large w_0 chosen, one needs a fitting radial polynomial of power N equal to at least $N = 16$ in order to get an accurate result for M_x^2, which otherwise can be exactly calculated by our simple analytical formula (15). In cases of smaller w_0, the required N for sufficient fitting of $\Phi(r)$ over a unit radius aperture will be even larger. Thus, we come to the conclusion that analytical study of self-induced beam

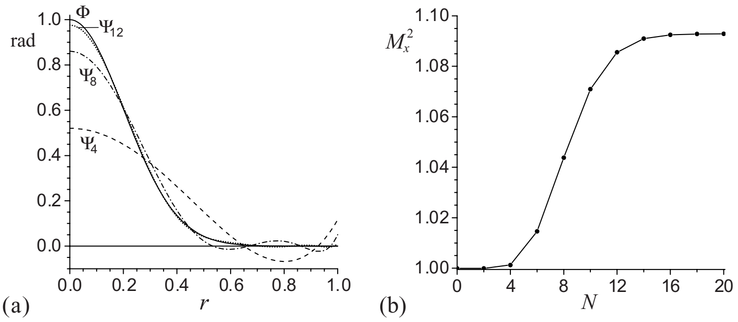

Figure 1. (*a*) Polynomial fits $\Psi_N(r)$ with $N = 4,8,12$ of phase profile $\Phi(r)$ with $\Phi_0 = 1$ of the self-phase modulated Gaussian beam with size $w_0 = 0.4$. (*b*) Convergence of beam quality parameter M_x^2 of amplitude profile $\rho(r)\cdot\exp(\mathrm{i}\Psi_N(r))$ with N; compare with equations (14) and (15).

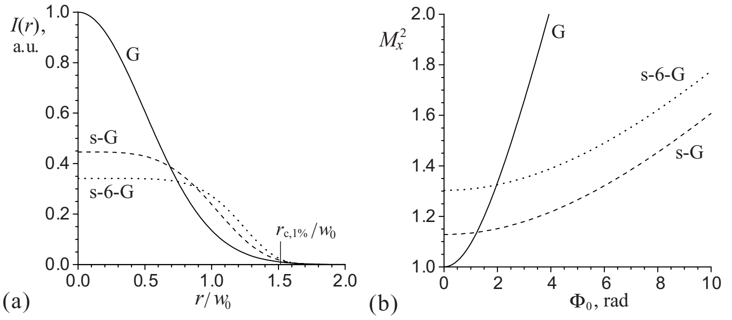

Figure 2. (*a*) Intensity profiles of beams with the same power and sizes adjusted by 1%-residual power criterion. (*b*) Deterioration of their M_x^2 with self-phase modulation, e.g. due to increasing of beam power; the argument Φ_0 is the central phase for Gaussian beam in equations (14) and (15).

aberrations should be performed in terms of localized mode profiles, see equation (16), without taking into consideration the size of the reference aperture usually used to study polynomial aberrations in imaging systems.

6. A self-phase modulated super-Gaussian beam

Now let us study the question of whether a Gaussian beam is the most suitable for free space optical power delivery in terms of the sensitivity of M_x^2 to self-phase modulation distortion occurring at the output of its generating laser system. Consider now the normalized so-called super-Gaussian profile $U_{\mathrm{sG}}(r) = (2/\pi)^{3/4}/u_0\cdot\exp(-r^4/u_0^4)$ at beam waist position. Calculations (15) for self-phase modulation of this profile give

$$U_{\mathrm{sG}}\cdot\exp\left(\mathrm{i}\Theta_0 \mathrm{e}^{-2\frac{r^4}{u_0^4}}\right) \;\to\; M_{x,\mathrm{sG}}^2 = \frac{2}{\sqrt{\pi}}\sqrt{1+\left(\frac{4}{9}-\frac{\pi}{8}\right)\Theta_0^2}\,. \tag{19}$$

Here Θ_0 is the central value of the phase distortion profile proportional to the intensity profile. A generalized analytical expression for arbitrary localized phase aberration similar to equation (16) can be obtained by introducing orthogonal modes based on a power series of r^4 with the common factor $\exp(-2r^4/u_0^4)$ of the intensity profile and performing calculations (15).

The resulting values of M_x^2 in equations (15) and (19) for both beam profiles depend on the central phases Φ_0 and Θ_0 which are proportional to the central intensities $|U_{\mathrm{G}}(0)|^2$ and $|U_{\mathrm{sG}}(0)|^2$, which on their own depend on the corresponding beam sizes w_0 and u_0. So, in order to compare self-phase modulation sensitivities we have to use certain criteria for the size ratio of their beam sizes. If both beams have the same power and have the same mean square average size $\langle x^2\rangle = \frac{1}{4}w_0^2 = u_0^2/(2^{3/2}\pi^{1/2})$ then they will have central phases caused by the same self-phase modulation process and are related to each other by $\Theta_0 = |U_{\mathrm{sG}}(0)|^2/|U_{\mathrm{G}}(0)|^2\cdot\Phi_0 = (2/\pi)^{1/2}(w_0/u_0)^2\cdot\Phi_0 = 2/\pi\cdot\Phi_0$.

The size u_0 of the super-Gaussian beam can be increased even more in comparison with w_0 because of a faster decrease of its profile at large r. In the case of a specific 1%-residual power criterion, when both beams outside the same radius $r_{\mathrm{c},1\%}$ have the same amount of power $0.01\cdot P$ the beam size ratio is equal $u_0/w_0 = 1.337$, $r_{\mathrm{c},1\%} = 1.517w_0$, which determines new phase relation $\Theta_0 = 0.446\cdot\Phi_0$ different from the previous one.

Figure 2(*a*) demonstrates this super-Gaussian profile relative to a Gaussian one of the same power with additional profile $U_{\mathrm{s\text{-}6\text{-}G}}(0)\cdot\exp(-r^6/s_0^6)$. We denoted it by 'super-6-Gaussian'. Widths of the profiles are mutually adjusted by the same 1%-residual power criterion. Figure 2(*b*) presents deterioration of M_x^2 for these three beam profiles due to the same self-phase modulation effect. The chosen argument is the central phase Φ_0 of the Gaussian beam, so the phase Θ_0 of the super-Gaussian profile in equation (19) is expressed through Φ_0 according to their last mutual relation presented above.

One can notice that for a particular value $M_x^2 = 1.5$, the super-Gaussian beam provides more than three times better tolerance for self-phase modulation parameter n_2 or Δz or beam power according to equation (14) despite its initially higher $M_x^2 = 2/\pi^{1/2} = 1.128$. Higher-order super-Gaussian beams with significantly higher initial M_x^2 do not provide a further advantage.

7. Conclusion

We have derived analytical expressions (15) and (19) for the propagation invariant beam quality parameter M_x^2 of self-phase modulated Gaussian and super-Gaussian beam profiles correspondingly. The result formulated for the Gaussian profile is applicable for self-phase modulation occurring at arbitrary position relative to its waist location due to propagation self-similarity of the Gaussian beam. Those expressions depend only on corresponding central phases of self-phase modulation. We also have discussed generalization of the derived analytical results in cases of arbitrary localized radial phase distortions. We have found that a beam with a super-Gaussian profile may be preferable in high-power laser applications than a Gaussian beam for a reasonable ratio of their sizes due to its greater resistance to deterioration of beam quality because of self-phase modulation. The obtained results can be applied to measurements of optical nonlinearity. The main technical advantage of our results presented in the form of relatively simple analytical expressions is that traditional approaches based on polynomial representation of aberrations cannot efficiently reproduce them due to the poor convergence of power series for Gaussian profiles. Finally, the presented results can be applied for fast performance evaluation of thermally distorted optical elements.

References

[1] Siegman A E 1990 New developments in laser resonators *Proc. SPIE* **1224** 2–14

[2] Johnston T F Jr 1990 M2 concept characterizes beam quality *Laser Focus World* **26** 173–83

[3] Belanger P A 1991 Beam propagation and the ABCD ray matrices *Opt. Lett.* **16** 196–8

[4] Bastiaans M J 1991 Second-order moments of the Wigner distribution function in first order optical systems *Optik* **88** 163–8

[5] Neubert B J and Eppich B 2005 Influences on the beam propagation ratio M2 *Opt. Commun.* **250** 241–51

[6] Peñano J, Sprangle P, Ting A, Fischer R, Hafizi B and Serafim P 2009 Optical quality of high-power laser beams in lenses *J. Opt. Soc. Am.* B **26** 503–10

[7] Jalali A, Rybarsyk J and Rogers E 2013 Thermal lensing analysis of TGG and its effect on beam quality *Opt. Express* **21** 13741–7

[8] Pan S Q, Ye Z B, Zhao Z G, Liu C, Ge J H, Xiang Z and Chen J 2013 Beam quality improvement by thermally induced aberrations in a diode-end-pumped laser amplifier *Laser Phys.* **23** 095003

[9] Sasián J 2013 *Introduction to Aberrations in Optical Imaging Systems* (Cambridge: Cambridge University Press)

[10] Siegman A E 1993 Analysis of laser beam quality degradation caused by quartic phase aberrations *Appl. Opt.* **32** 5893–901

[11] Alda J, Alonso J and Bernabeu E 1997 Characterization of aberrated laser beams *J. Opt. Soc. Am.* A **14** 2737–47

[12] Mafusire C and Forbes A 2011 Generalized beam quality factor of aberrated truncated Gaussian laser beams *J. Opt. Soc. Am.* A **28** 1372–8

[13] Hong H L, Liu Q, Huang L and Gong M L 2013 Numerical and experimental results of wavefront aberration caused in a high power laser amplifier *Laser Phys. Lett.* **10** 025002

[14] Qiu Y, Gong M, Huang L, Liu Q, Yan P, Zhang H and Li T 2013 The beam quality of a truncated Gaussian beam with aberrations *Laser Phys. Lett.* **10** 055001

[15] Siegman A E 1986 *Lasers* (Sausalito, CA: University Science Books) p 633

[16] Serna J, Martínez-Herrero R and Mejías P M 1991 Parametric characterization of general partially coherent beams propagating through ABCD optical systems *J. Opt. Soc. Am.* A **8** 1094–8

[17] Sheik-Bahae M, Said A A, Wei T-H, Hagan D J and Van Stryland E W 1990 Sensitive measurement of optical nonlinearities using a single beam *IEEE J. Quantum Electron.* **26** 760–9

[18] Sevian A, Andrusyak O, Ciapurin I, Smirnov V, Venus G and Glebov L 2008 Efficient power scaling of laser radiation by spectral beam combining *Opt. Lett.* **33** 384–6

[19] Wirth C *et al* 2011 High average power spectral beam combining of four fiber amplifiers to 8.2 kW *Opt. Lett.* **36** 3118–20

[20] Martinez-Herrero R and Mejias P M 1998 Truncation error of the Laguerre–Gauss expansion of axially symmetric beams in terms of second-order intensity moments *Pure Appl. Opt.* **7** 1231–6